Abstract. In this paper we present a novel algorithm, referred to as the non-linear intrinsic reconstitution algorithm, for three-dimensional structure determination of large biological molecules from cryo-electron microscopy projection images (cryo-EM for short), focusing our attention on the situation where the molecule admits non-trivial symmetries. Our algorithm constitutes a far reaching generalization of the intrinsic reconstitution algorithm presented in [13], which does not account for molecules with non-trivial symmetry. The formal justification of the algorithmic procedure is based on studying the spectrum of various integral operators, related to parallel transportation on the two-dimensional sphere, called invariant transport operators. In this regard, the main technical result of this paper is a complete description of the spectrum of the invariant transport operators, generalizing earlier results presented in [11] and [12]. Along the way, we continue to develop the mathematical foundations of three-dimensional cryo-EM, further elucidating the central role played by representation theoretic principles in this scientific discipline.

0. Introduction

Symmetric patterns are omnipresent in almost every scientific discipline. In the realm of molecular biology, point symmetries are governing the structure of many important large molecules such as various proteins and exterior shells of viruses. Symmetric biological molecules, usually appear as complexes, composed of various physical transformations of a single amorphous unit, hence the classification of their symmetry corresponds to the mathematical problem of classifying finite groups of three-dimensional rotations. The solution of the latter mathematical problem is well known: there are finite number of symmetry types: the trivial group, cyclic groups, dihedral groups and three sporadic types corresponding to the symmetries of the Platonic solids, namely, symmetries of the Tetrahedron, the Octahedron and the Icosahedron. Interestingly, there are examples of large molecules of biological origin admitting every type of finite symmetry in this list (see Figure 1). In this paper, we investigate the mathematical and algorithmic aspects involved with the presence of non-trivial point symmetries in the context of three-dimensional structure determination of large biological molecules using cryo-electron microscopy.

Three-dimensional structure determination of large biological molecules is a central problem in structural biology, as witnessed, for example, by the 2003 Chemistry Nobel Prize, co-awarded to R. MacKinnon for resolving the three-dimensional structure of the Shaker K+ channel protein [1, 8], and by the 2009 Chemistry Nobel Prize, awarded to V. Ramakrishnan, T. Steitz and A. Yonath for studies of

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Figure 1. Symmetric macromolecular complexes that appear in nature. These complexes correspond to the various finite groups of rotations: the first column consists of complexes with cyclic symmetry, the second column consists of complexes with dihedral symmetry and the third column consists of complexes with the symmetries of Platonic solids - Tetrahedral (T), Octahedral (O) and Icosahedral (I).

the structure and function of the ribosome. Cryo-electron microscopy (cryo-EM for short) is a promising approach to three-dimensional structure determination of large molecules, representing, an alternative to X-ray crystallography. The challenge in this latter method is often more in the crystallization itself than in the interpretation of the X-ray results, since many large molecules have so far withstood all attempts to crystallize them. In cryo-EM, the three-dimensional structure is determined from noisy projection images taken at unknown random orientations by an electron microscope, i.e., a random Computational Tomography (CT). More specifically, samples of identical molecules are rapidly immobilized in thin layer of vitreous ice (this is an ice without crystals). The cryo-EM imaging process produces a large collection of tomographic projections, corresponding to many copies of the same molecule, each immobilized in a different (yet unknown) orientation, where the intensity of the pixels in a given projection image is correlated with the integrals of the electric potential along the path of imaging electrons (see Figure 2). The goal is to reconstruct the three-dimensional structure of the molecule from such a collection of projection images. There are two main difficulties involved:
the first difficulty is that the highly intense electron beam destroys the molecule and it is therefore impractical to take projection images of the same molecule at known different directions as in the case of classical CT, hence the orientation of the molecule that produces every image is unknown. The second difficulty concerns the extremely low signal-to-noise ratio (SNR) of every projection image, mostly due to shot noise induced by the maximal allowed electron dose, since a single copy of the molecule can be imaged only once.

0.1. Mathematical model of cryo-EM. Instead of thinking of a multitude of molecules immobilized in different orientations and observed by an electron microscope held in a fixed position, it is more convenient to think of a single molecule, observed by an electron microscope from different orientations. Thus, the orientation describes the configuration of the microscope instead of that of the molecule. In order to specify the mathematical model, we require the following terminology:

Let \((V,r)\) be an oriented three-dimensional Euclidean vector space. The reader can take \(V\) to be \(\mathbb{R}^3\) and \(r\) to be the standard inner product. Let \(X = \text{Fr}(V)\) be the oriented frame manifold associated to \(V\). A point \(x \in X\) is a map \(x: \mathbb{R}^3 \to V\) satisfying \(x^T \circ x = Id\) (orthogonal map). A Frame \(x: \mathbb{R}^3 \to V\) corresponds in a one-to-one fashion to an orthonormal basis \((e_1,e_2,e_3)\) of \(V\) compatible with the orientation, where \(e_1 = x(1,0,0)\), \(e_2 = x(0,1,0)\) and \(e_3 = x(0,0,1)\). The third vector \(e_3\) is distinguished, referred to as the viewing direction of the frame and denoted by \(\pi(x)\). Using this terminology, the physics of cryo-EM is modeled as follows:

- The molecule is modeled by a real valued function \(\phi: V \to \mathbb{R}\), describing the electromagnetic potential induced by the molecule.
- An orientation of the microscope is modeled by a frame \(x \in X\). The third vector \(\pi(x)\) is the viewing direction of the microscope and the plane
Figure 3. A frame \( x = (e_1, e_2, e_3) \), modeling the orientation of the electron microscope, where the vector \( e_3 \) is the viewing direction and the vectors \( e_1, e_2 \) establish the coordinates of the camera.

spanned by the first two vectors \( e_1 \) and \( e_2 \) is the plane of the camera equipped with the coordinate system of the camera (see Figure 3).

- The projection image obtained by the microscope, when observing the molecule from an orientation \( x \) is a real valued function \( I : \mathbb{R}^2 \to \mathbb{R} \), given by the X-ray projection along the viewing direction:

\[
I(p, q) = \text{X-ray}_x(\phi)(p, q) = \int_0^1 \phi(\lambda(p e_1 + q e_2 + t e_3)) \, dt.
\]

The data collected from the experiment is a finite set, consisting of \( N \) projection images \( P = \{I_1, ..., I_N\} \), where each image in this set corresponds to a different (yet unknown) orientation of the electron microscope.

0.2. Main problem. Roughly speaking, the main problem of cryo-EM is to compute the orientation associated with every projection image. In order to give a precise formulation of this problem we require some additional terminology. An orthogonal similitude of \( V \) is a linear transformation \( T : V \to V \) that preserves the inner product \( r \) up-to a positive constant, namely, \( r(Tu, Tv) = \lambda \cdot r(u, v) \), for every \( u, v \in V \) and for some \( \lambda \in \mathbb{R}^{>0} \). The collection of all orthogonal similitudes forms a group, which is denoted by \( GO(V) \); an element in this group can be thought of as a composition of an orthogonal transformation (not necessarily preserving orientation) and scaling.

0.2.1. Frame reconstruction problem. The simpler scenario, arising when the molecule admits no non-trivial symmetry, is when the potential function \( \phi \) is generic, in the sense that each projection image \( I_i \in P \) corresponds to a unique frame

\[
x_i \in X.
\]
In this situation, the problem is to compute the frame associated with every projection image up to a global action of an orthogonal similitude. More concretely, the goal is to compute a collection of maps $y_1, \ldots, y_N : \mathbb{R}^3 \to V$ such that $g \circ y_i = x_i$ for some fixed element $g \in GO(V)$ (the element $g$ does not depend on the index $i$).

Using the maps $y_1, \ldots, y_N$, every image can be placed in its appropriate place in $V$ and the potential $\phi$ can be reconstructed using the inverse X-ray transform, up to an action of an orthogonal similitude. We remark, that according to this reconstruction we loose track of the handedness of the molecule and its scale, however, these should not be considered as severe restrictions, since both parameters are known in advance for molecules of biological origin.

0.2.2. **Orbit reconstruction problem.** The more complicated scenario, arising when the molecule is symmetric, is when the potential $\phi$ is invariant under a non-trivial finite subgroup $\Gamma \subset SO(V)$, namely

$\phi(\gamma \cdot v) = \phi(v),$

for every $\gamma \in \Gamma$ and $v \in V$. The complication arising in this situation is that each projection image $I_i \in \mathcal{P}$ corresponds to a $\Gamma$-orbit of frames $\pi_i \in \Gamma \backslash X$, due to the fact that projection images associated with frames lying in the same $\Gamma$-orbit cannot be distinguished. Consequently, the problem now becomes to compute the orbit associated with each projection image, up to an orthogonal similitude. In more details, the goal is to compute a collection of maps $y_1, \ldots, y_N : \mathbb{R}^3 \to V$ such that $g \circ y_i$ lies in the orbit $\pi_i$, for some fixed element $g \in N'(\Gamma)$, where $N'(\Gamma)$ is the normalizer of $\Gamma$ inside $GO(V)$.

We remark that in the presence of a non-trivial group of symmetries, every projection image yields more "X-ray" information about the function $\phi$, since every single image $I \in \mathcal{P}$ can be placed with accordance to $n = |\Gamma|$ orientations - each associated with a different frame in the corresponding orbit. For this reason, presence of symmetries is considered a favorable scenario for a faithful three-dimensional cryo-EM reconstruction. However, the mathematical set-up is considerably more sophisticated compared to the generic situation, as the structure of the orbit space $\Gamma \backslash X$ is more complicated than that of the frame manifold $X$.

**Remark 0.1.** Two remarks are in order at this point. First remark is that the reconstruction problem is non-linear and, is furthermore non-convex. This is apparent already in case of the frame reconstruction problem, as the orthogonality condition defining a frame is a non-convex condition, and even more extremely so, in case of the orbit reconstruction problem when a non-trivial group of symmetries is present. The immediate consequence is, that standard convex optimization do not apply very well to our set-up. Second remark is that the projection images collected from the experiment are extremely noisy, which forces the reconstruction algorithm to work under very low SNR conditions.

Another computational problem in this context is to compute from the set of projection images the symmetry group $\Gamma$. We note, that its not realistic in practice to determine this group just from looking at specific projection images (see Figure 4). Instead, one seeks a reliable algorithmic procedure that determines the symmetry group of the molecule from the set of projection images in a "global" fashion.
Figure 4. The first line consists of clean simulated projection images of an asymmetric complex of E.coli bound to telithromycin. The second line consists of clean simulated projection images of a C2 symmetric complex of the Human transferrin receptor. The images in each column corresponds to the same orientation of the electron microscope.

0.3. Common line data. A preliminary step for all reconstruction algorithms is to compute from the set of projection images a certain geometric data, called common line data. The common line data is a set consisting of ordered pairs of unit vectors in $\mathbb{R}^2$, called common line pairs. Each common line pair is associated with a different ordered pair of projection images.

**Definition 0.2.** A common line pair associated with projection images $I_i, I_j \in \mathcal{P}$ is an ordered pair of unit vectors $(C_{ij}, C_{ji}) \in \mathbb{R}^2 \times \mathbb{R}^2$, characterized by the condition

\[(0.1) \quad \mathcal{F}(I_i)(\Re C_{ij}) = \mathcal{F}(I_j)(\Re C_{ji}),\]

where $\mathcal{F}$ stands for the Fourier transform on $\mathbb{R}^2$.

In order to understand the geometric meaning of the above definition, we use a basic property of the Fourier transform, called the Fourier slicing property. To express this property, we require the following terminology: given a frame $x \in X$, we define its principal part to be the orthogonal map $p(x) : \mathbb{R}^2 \rightarrow V$, obtained by restricting $x$ to the plane spanned by the vectors $(1,0,0)$ and $(0,1,0)$. The Fourier slicing property asserts that

\[(0.2) \quad \mathcal{F}(\phi) \circ p(x) = \mathcal{F}(X\text{-ray}_x(\phi)),\]

for every frame $x \in X$, where $\mathcal{F}$ on the left and on the right stands for the Euclidean Fourier transform on $V$ and the Fourier transform on $\mathbb{R}^2$ respectively. The reader can easily convince himself that (0.2) amounts to the the elementary fact that the Fourier transform interchanges the operation of restriction with the operation of integration. Consequently, given projection images

\[
I_i = X\text{-ray}_{x_i}(\phi),
\]

\[
I_j = X\text{-ray}_{x_j}(\phi),
\]
the Fourier slicing property implies the following relations
\[ \mathcal{F}(I_i) = \mathcal{F}(\phi) \circ p(x_i), \]
\[ \mathcal{F}(I_j) = \mathcal{F}(\phi) \circ p(x_j). \]
Assuming, we are in the generic situation where each projection image is associated with a unique frame, the above relations suggests that the common line vectors \( C_{ij} \) and \( C_{ji} \) satisfy the condition
\[ p(x_i) (C_{ij}) = p(x_j) (C_{ji}). \]
If we denote by \( L \) the line of intersection (common line) between the subplanes \( \text{Im} \, p(x_i) \subset V \) and \( \text{Im} \, p(x_j) \subset V \), then, based on (0.3), we can use the common line pair \( (C_{ij}, C_{ji}) \) to express the canonical identification of the two coordinate realizations of \( L \), given by the matrix
\[ C(x_i, x_j) = p(x_i)^T \circ P_L \circ p(x_j), \]
where \( P_L : V \to V \) is the orthogonal projection on the common line \( L \). Specifically, we have the following formula
\[ C(x_i, x_j) = C_{ij} \cdot C_{ji}^T. \]
In the above formula, the vectors \( C_{ij} \) and \( C_{ji} \) are considered as column vectors, hence \( C_{ji}^T \) is a row vector. To summarize, we observe that although we do not know the frame \( x_i \) associated with a projection image \( I_i \), we still are able to extract for every pair of projection images \( I_i \) and \( I_j \), certain geometric information relating the corresponding frames \( x_i \) and \( x_j \), in the form of the matrix \( C(x_i, x_j) \).

**Remark 0.3.** In practice, instead of looking for equality in (0.1), one looks for high correlation of the corresponding one dimensional signals.

The situation when the potential \( \phi \) is invariant under a finite subgroup \( \Gamma \) is slightly more involved. In this situation there corresponds a finite collection common line pairs, with every ordered pair of projection images. To see the geometry behind this phenomena, let us assume that the projection images \( I_i \) and \( I_j \) corresponds to the \( \Gamma \)-orbits \( \pi_i \) and \( \pi_j \) respectively. Under this assumption, we can think of the images \( I_i \) and \( I_j \) as corresponding to every choice of frames \( x_i' \in \pi_i \) and \( x_j' \in \pi_j \) in the corresponding orbits. Hence, for every such choice, by (0.2), we have the relation
\[ \mathcal{F}(I_i) = \mathcal{F}(\phi) \circ p(x_i'), \]
\[ \mathcal{F}(I_j) = \mathcal{F}(\phi) \circ p(x_j'). \]
which, in turns, determines a common line pair \( (C_{ij}, C_{ji}) \in \mathbb{R}^2 \times \mathbb{R}^2 \). Further inspection reveals that any other representative \( (x''_i, x''_j) \) satisfying \( (x''_i, x''_j) = (\gamma \circ x'_i, \gamma \circ x'_j) \), for some element \( \gamma \in \Gamma \), determines the same common line pair (perhaps, up to a minus sign). According to this reasoning, it is natural to index common line pairs by \( \Gamma \)-orbits in the Cartesian product set \( \pi_i \times \pi_j \), where \( \Gamma \) acts diagonally. Since the number of such orbits is equal the number of elements in the group \( \Gamma \), it follows that each ordered pair of projection images determines a collection of common line pairs \( (C^k_{ij}, C^k_{ji}) \in \mathbb{R}^2 \times \mathbb{R}^2 \), where \( k = 1, \ldots, |\Gamma| \). Furthermore, we note that there is no way to distinguish a special element in this collection and all these pairs should be treated on an equal footing (see Figure 5).
Figure 5. The left landscape is a typical common line profile of the C2 symmetric complex of the Human transferrin receptor. The right landscape is a typical common line profile of the asymmetric complex of E.coli bound to telithromycin. The profiles for both molecules are between a pair of simulated projection images corresponding to the same pair of the electron microscope orientations. The first horizontal axis measures the angle of the unit vector in the plane of the first image and the second horizontal axis measures the angle of the the unit vector in the plane of the second image. The vertical axis measures the strength of the correlation between the Fourier transforms of the images at a specific pair of vectors. The peaks appear over every pair of common line vectors $(C_{ij}, C_{ji})$ and over its antipode $(-C_{ij}, -C_{ji})$.

0.4. The intrinsic reconstitution algorithm. An algorithm, referred to as the intrinsic reconstitution algorithm, for solving the frame reconstruction problem, under the condition that the points $x_1, \ldots, x_N \in X$ are distributed independently and uniformly at random, was presented in [13]. The appealing properties of the algorithm are its remarkable numerical stability to noise and its efficient running time. The mathematical theory behind the algorithm was explained in [11] and used in order to provide a conceptual explanation of its admissibility (correctness) and a proof of its numerical stability. We proceed to describe the algorithmic procedure and its mathematical theory, serving as a starting point for the explanation of the main ideas and results of the present paper.

0.4.1. Input and output of the algorithm. The first step is to represent each frame $x \in X$ by its principal part $p(x)$, noting that no information is lost in this translation as every frame can be uniquely reconstructed from its principal part according to the rule:

$$
e_1 = p(x) (1, 0, 0),$$
$$
e_2 = p(x) (0, 1, 0),$$
$$
e_3 = e_1 \times e_2,$$

where $\times$ stands for the operation of vector product. and to forget the orientation of the vector space $V$. Note that knowing the principal parts is sufficient in order to place every projection image in its appropriate place in $V$. We proceed to describe the input and output of the algorithm.
The input of the algorithm is the common line data \( \{(C_{ij}, C_{ji}) : i \neq j\} \). The output of the algorithm is a collection of maps \( p_i : \mathbb{R}^2 \to V, i = 1, \ldots, N \), such that \( g \circ p_i = p(x_i) \), for some fixed element \( g \in GO(V) \). We note, that the algorithm computes only the principal parts of the corresponding frames, nevertheless, this is sufficient in order to place every image in its appropriate location.

0.4.2. Body of the algorithm. The idea of the algorithm is to construct an intrinsic model \( \hat{V} \) of the Euclidean vector space \( V \), that is defined solely in terms of the common line data and, then, to solve the frame reconstruction problem in the coordinates of this new vector space.

The algorithm proceeds in four steps:

1. Consider the Euclidean vector space 
   \( \mathbb{R}^{2N} = \mathbb{R}^2 \times \ldots \times \mathbb{R}^2 \), \( N \) times.

2. Define the \( 2N \times 2N \) matrix \( C : \mathbb{R}^{2N} \to \mathbb{R}^{2N} \), composed of \( N \times N \) blocks, where the \((i, j)\) block is the \( 2 \times 2 \) matrix:
   \[
   C(i, j) = \frac{1}{N} C_{ij} \cdot C_{ji}^T,
   \]
   when \( i \neq j \) and it is the zero matrix when \( i = j \). The matrix \( C \) is symmetric since, evidently, \( C(i, j) = C(j, i)^T \), for every \( 1 \leq i, j \leq N \).

3. Since the matrix \( C \) is symmetric, it induces a spectral decomposition
   \[
   \mathbb{R}^{2N} = \bigoplus_{\lambda} \left( \mathbb{R}^{2N} \right)_\lambda.
   \]

Define the vector space
   \[
   \hat{V} = \bigoplus_{\lambda > 1/3} \left( \mathbb{R}^{2N} \right)_\lambda.
   \]

Let us denote by \( pr_i : \hat{V} \to \mathbb{R}^2 \) the standard orthogonal projection on the \( i \)th component. Since both domain and range are Euclidean vector spaces, \( pr_i \) admits a transpose, which we denote by \( \varphi_i : \mathbb{R}^2 \to \hat{V} \).

Remark 0.4. We note, that in practice, one takes \( \hat{V} \) to be the subspace spanned by the eigenvectors of \( C \) associated with the first three maximal eigenvalues.

0.4.3. Formal justification of the algorithm. The admissibility (correctness) of the algorithmic procedure, or in other words, the sense by which this output data establishes a solution to the frame reconstruction problem is encapsulated in the content of the following claim:

Claim 0.5. There exists a linear map \( \tau : V \to \hat{V} \) satisfying \( \tau^T \circ \tau = \lambda \), for some \( \lambda \in \mathbb{R} > 0 \) (scaled isometry) and having the property that
   \[
   \varphi_i = \tau \circ p(x_i),
   \]
   for every \( i = 1, \ldots, N \). Consequently, by choosing an arbitrary isometry \( \tau' : \hat{V} \to V \), the maps \( p_i = \tau' \circ \varphi_i \) establish a solution to the frame reconstruction problem.
Claim 0.5 is justified by studying the continuous limit when the number of images goes to infinity. One observes that, in the limit, the finite collection of points \( x_1, \ldots, x_N \) approximates the frame manifold \( X \) equipped with the unique normalized Haar measure \( \mu \). This assertion follows by the assumption about the uniform distribution of the points \( x_1, \ldots, x_N \). As a consequence, the finite dimensional vector space \( \mathbb{R}^{2N} \) approximates the vector space \( \mathbb{R}^2(X) \), consisting of \( \mathbb{R}^2 \)-valued functions on \( X \) and, moreover, the finite matrix \( C \) approximates the common line integral operator \( C \), given by

\[
C(f)(x) = \int_{y \in X} C(x, y) f(y) \, \mu(y),
\]

for every \( f \in \mathbb{R}^2(X) \). The following theorem provides a closed formula for the eigenvalues of the common line operator and their multiplicities.

**Theorem 0.6** (Spectral Theorem [11]). The common line operator \( C \) admits a kernel and a discrete spectrum \( \lambda_n, n \in \mathbb{N}_{\geq 1} \), where

\[
\lambda_n = \frac{(-1)^{n-1}}{n(n+1)}.
\]

Moreover, the multiplicity of \( \lambda_n \) is equal \( 2n + 1 \).

Theorem 0.6 implies that the maximal eigenvalue of the common line operator is \( \lambda_1 = 1/2 \) and its multiplicity is equal three and, moreover, there exists of a spectral gap, equal to \( \lambda_1 - \lambda_3 = 5/12 \), separating the maximal eigenvalue from the rest of the spectrum. Let us denote by \( \tau : V \to \mathbb{R}^2(X) \) the canonical map, given by

\[
\tau(v)(x) = p(x)^T(v),
\]

for every vector \( v \in V \) and a frame \( x \in X \). Claim 0.5 is a consequence of the following theorem:

**Theorem 0.7** (Admissibility theorem [11]). Let \( V \) denote the maximal eigenspace of the common line operator \( C \). The vector space \( V \) coincides with the image of the map \( \tau \) and, moreover, \( \tau : V \to V \) is an isometry up-to scaling.

Roughly speaking, the idea of the proof is to identify \( V \) with the unique copy of the three-dimensional irreducible representation of \( SO(V) \) lying inside \( \mathbb{R}^2(X) \) and to notice that \( \tau \) is a morphism of \( SO(V) \)-representations.

0.4.4. **Abstract layout of the algorithm.** The intrinsic reconstitution algorithm incorporates the following structures. A relaxation rule that represents each frame \( x \in X \) by its principal part \( p(x) \). A fundamental vector space, which is a distinguished three-dimensional subspace

\[
V \subset \mathbb{R}^2(X),
\]

defined in terms of the relaxation rule \( x \mapsto p(x) \), via the map \( \tau \). The common line operator, which is an integral operator

\[
C : \mathbb{R}^2(X) \to \mathbb{R}^2(X),
\]

whose maximal eigenspace is \( V \). And, finally, an approximation scheme, which is a procedure to compute the matrix \( C(x_i, x_j) \) from the corresponding projection images \( I_i \) and \( I_j \). These structures are incorporated in the following manner. The
relaxation rule is used to define the solution of the computational problem as a collection of maps
\[ p_1, \ldots, p_N : \mathbb{R}^2 \to V, \]
The core of the algorithmic procedure is to compute (an approximation of) the fundamental vector space \( \mathcal{V} \) incorporating its spectral characterization as the maximal eigenspace of the common line operator: the computation uses the approximation scheme in order to compute a numerical approximation of the kernel function of the common line operator from the projection images. The admissibility of the algorithm amounts to the fact that the map \( \tau : \mathcal{V} \to \mathcal{V} \) is an isometry up-to scaling. Finally, the numerical stability of the algorithm amounts to the fact that the common line operator admits a spectral gap.

0.5. Main results. The main contribution of the present paper is a non-linear generalization of the intrinsic reconstitution algorithm, referred to as the non-linear intrinsic reconstitution algorithm. While maintaining the appealing properties of numerical stability and efficient running time, our algorithm has the advantage of being powerful enough to account also for the reconstruction of general orbits. Part of the algorithm is is a numerically stable procedure for computing the symmetry group \( \Gamma \) from the set of projection images. The main elaboration over the basic algorithm is that the Euclidean vector space \( \mathcal{V} \subset \mathbb{R}^2(X) \) is replaced by a "richer" structure of a graded real algebra \( \mathcal{A} (\Gamma) \subset \mathbb{C}(\Gamma \setminus X) \),
\[ \mathcal{A} (\Gamma) = \bigoplus_{m \geq 0} \mathcal{A}_m (\Gamma). \]
The algorithm exploits the fact that the algebra \( \mathcal{A} (\Gamma) \) admits a spectral characterization, in the sense that each graded component \( \mathcal{A}_m (\Gamma) \), \( m \geq 1 \), coincides with maximal eigenspace of a specifically designed integral operator
\[ T_m (\Gamma) : \mathbb{C}(\Gamma \setminus X) \to \mathbb{C}(\Gamma \setminus X), \]
generalizing the common line operator. The admissibility (correctness) of the algorithm amounts to the existence of a canonical isomorphism of real graded algebras
\[ \tau : \mathcal{A} (\Gamma) \to \mathcal{A} (\Gamma), \]
where \( \mathcal{A} (\Gamma) \) is the \( \Gamma \)-invariant portion of a certain quotient of the polynomial algebra \( \text{Poly}(\mathcal{C}V) \). In addition, the numerical stability of the algorithm amounts to the fact that each of the operators \( T_m (\Gamma) \) admits a spectral gap that separates the maximal eigenvalue from the rest of the spectrum. With regard to the admissibility aspect, the main technical result of this paper is that the algebra \( \mathcal{A} (\Gamma) \) admits no automorphisms except the obvious geometric ones. This assertion is summarized in the following theorem:

**Theorem 0.8 (Rigidity property).** Let \( N' (\Gamma) \) denote the normalizer of the group \( \Gamma \) inside \( GO(V) \). The automorphism group of the real algebra \( \mathcal{A} (\Gamma) \) satisfy
\[ \text{Aut}_\mathbb{R} (\mathcal{A} (\Gamma)) = N' (\Gamma) / \Gamma. \]

**Remark 0.9.** the rigidity property is somewhat surprising when contrasted with the well known fact that the algebra of invariants with respect to a finite reflection subgroup in \( O(V) \) is a free algebra with two generators, therefore, admitting many automorphisms. Curiously, the symmetries which are relevant to three-dimensional
cryo-EM must reside in $SO(V)$, as explained in the first paragraph of the introduction.

With regards to the numerical stability aspect, the main technical result of this paper is a complete description of the spectrum of the various integral operators $T_m = T_m(\Gamma)$, in case $\Gamma$ is the trivial subgroup. This is summarized in the following generalization of Theorem 1.5:

**Theorem 0.10 (Generalized spectral theorem).** The operator $T_m$, $m \geq 1$, admits a kernel and a discrete real spectrum $\lambda_{n,m} \in \mathbb{R}$, $n \geq m$, such that

$$
\lambda_{n,m} = \frac{(-1)^{n-m} m}{n(n+1)}.
$$

Moreover, the multiplicity of the eigenvalue $\lambda_{n,m}$ is $2n + 1$.

Along the way, we generalize the representation theoretic development that was initiated in [11] and [12], thus further elucidating the central role played by representation theoretic principles in the field of three-dimensional cryo-electron microscopy. We devote the remainder of the introduction to a more detailed explanation of the main ideas and results underlying the non-linear intrinsic reconstitution algorithm.

**0.6. Relaxation rule.** The starting point is to introduce a relaxation rule, powerful enough for representing by linear algebra data orbits with respect to arbitrary finite subgroups $\Gamma \subset SO(V)$. Let us assume first that $\Gamma$ is the trivial subgroup. In this situation, a frame $x \in X$ is represented by an algebra character $\delta_x: A = \text{Poly}(CV) / (r) \rightarrow \mathbb{C}$, where the Euclidean metric $r$ is considered as a quadratic complex polynomial. The algebra character $\delta_x: A \rightarrow \mathbb{C}$ is defined by evaluating a polynomial on the vector $e_1 - ie_2 \in CV$. We note, that no information is lost in this representation, as every frame can be uniquely reconstructed from its corresponding character, however, notice that characters associated to actual frames form a strict subset of the set of all characters.

The rule $x \mapsto \delta_x$ should be considered as a non-linear extension of the rule $x \mapsto p(x)$, since the restriction $\delta_x : CV \rightarrow \mathbb{C}$ is the unique complex extension of the map $p(x)^\Gamma : V \rightarrow \mathbb{R}^2 = \mathbb{C}$. The advantage of the extended rule is that it can be naturally generalized to represent orbits with respect to arbitrary finite subgroups. In this more general situation, an orbit $x \in X$ is represented by an algebra character $\delta_x: A(\Gamma) \rightarrow \mathbb{C}$, where $A(\Gamma) = \mathbb{A}^\Gamma$ is the subalgebra of $\Gamma$-invariant polynomials in $A$. The algebra $A(\Gamma)$ is a graded real algebra:

$$
A(\Gamma) = \bigoplus_{m \geq 0} A_m(\Gamma),
$$

where the $m$th graded component $A_m(\Gamma)$ consists of invariant polynomials of degree $m$ and the real structure is induced from the operation of complex conjugation. Finally, the algebra $A(\Gamma)$ is rigid, in the sense that it admits no automorphisms except of the obvious geometric ones, as summarized in Theorem 0.8.
0.7. The fundamental algebra. The relaxation rule \( \pi \mapsto \delta_\pi \) defines a distinguished graded real subalgebra \( \mathcal{A}(\Gamma) \subset \mathbb{C}(\Gamma \setminus X) \), called the fundamental algebra. The fundamental algebra is defined as the image of the algebra map \( \tau : A(\Gamma) \to \mathbb{C}(\Gamma \setminus X) \), which is induced from the relaxation rule according to the formula

\[
\tau(a)(\pi) = \delta_\pi(a),
\]

for every \( a \in A \) and \( \pi \in \Gamma \setminus X \). The 0th graded component \( \mathcal{A}_0(\Gamma) \) is defined to be the image of \( A_0(\Gamma) \) under the map \( \tau \). The definition of the real structure is postponed to the body of the paper. The bottom line is that the map

\[
\tau : A(\Gamma) \to \mathcal{A}(\Gamma),
\]

is an isomorphism of graded real algebras. The fundamental algebra should be considered as a generalization of the fundamental vector space, since, in case \( \Gamma \) is the trivial subgroup, \( \mathcal{A}_1 = \mathcal{A}_1(\Gamma) \) coincides with the complex spanning of \( V \subset \mathbb{R}^2(X) \).

0.8. The transport operators. The main attribute of the fundamental algebra is that it admits a stable spectral characterization, in the sense that, each graded component \( \mathcal{A}_m(\Gamma) \), \( m \geq 1 \), coincides with the maximal eigenspace of an integral operator

\[
\mathcal{T}_m(\Gamma) : \mathbb{C}(\Gamma \setminus X) \to \mathbb{C}(\Gamma \setminus X),
\]

called the \( \Gamma \)-invariant transport operator of level \( m \). Where, stability amounts to the fact that there exists a spectral gap that separates the corresponding maximal eigenvalue from the rest of the spectrum. In case \( \Gamma \) is the trivial subgroup, the kernel function of \( \mathcal{T}_1 = \mathcal{A}_1(\Gamma) \) coincides with the complex spanning of \( V \subset \mathbb{R}^2(X) \).

(0.7) \[ \mathcal{T}_m(x, y) = \mathcal{T}_1(x, y)^m. \]

In this respect the \( \Gamma \)-invariant transport operators should be considered as generalizations of the common line operator. A complete description of the spectrum of \( \mathcal{T}_m \) can be obtained using techniques from representation theory and is summarized in Theorem 0.10.

Remark 0.11. The reasoning behind the name of the transport operator is the fact that the matrix \( \mathcal{T}_1(x, y) \) can be characterized in terms of parallel transportation on the two-dimensional sphere. A more comprehensive discussion of this aspect of the theory appears in [12] and [14] in the context of the class averaging problem.

In case \( \Gamma \) is an arbitrary finite subgroup, the operator \( \mathcal{T}_m(\Gamma) \) can be identified with the restriction of \( \mathcal{T}_m \) to the subspace of \( \Gamma \)-invariant functions on \( X \), that is

(0.8) \[ \mathcal{T}_m(\Gamma) = \mathcal{T}_m|_{\mathcal{C}(X)^\Gamma}, \]

implying that the spectrum of \( \mathcal{T}_m(\Gamma) \) is obtained from the spectrum \( \mathcal{T}_m \) by taking \( \Gamma \)-invariants. Another implication of (0.8) is a formula for the kernel function of \( \mathcal{T}_m(\Gamma) \) expressed in terms of the kernel function of \( \mathcal{T}_m \), which reads as follows:

(0.9) \[ \mathcal{T}_m(\Gamma)(\bar{x}, \bar{y}) = |\Gamma|^{-2} \sum_{x' \in \pi} \sum_{y' \in \bar{y}} \mathcal{T}_m(x', y'). \]
0.9. Approximation scheme. We proceed to describe an approximation scheme for computing the kernel functions of the various transport operators from the projection images. This procedure should be considered as a generalization of the approximation scheme for computing the kernel function of the common line operator. Let us assume first that $\Gamma$ is the trivial subgroup. In this situation, the matrix $T_1(x_i, x_j)$ can be expressed, using (0.6), in terms of the common line pair $(C_{ij}, C_{ji})$ according to the formula

$$T_1(x_i, x_j) = C_{ij} \cdot C_{ji}^T + \iota \cdot C_{ij} \cdot C_{ji}^\iota \cdot \iota^{-1}.$$ 

Alternatively, the reader can easily verify that $T_1(x_i, x_j)$ can be characterized as the unique rotation that sends the vector $C_{ji}$ to the vector $C_{ij}$. Consequently, using (0.7), the matrix $T_m(x_i, x_j)$ can be expressed in terms of $(C_{ij}, C_{ji})$ according to the formula

$$T_m(x_i, x_j) = [C_{ij} \cdot C_{ji}^T + \iota \cdot C_{ij} \cdot C_{ji}^\iota \cdot \iota^{-1}]^m.$$ 

In the situation when $\Gamma$ is an arbitrary finite subgroup, the various $\Gamma$-invariant transport operators can be expressed from the collection of common line pairs $(C_{ij}^k, C_{ji}^k)$, $k = 1, \ldots, |\Gamma|$, using Formula (0.9). The precise formula is

$$T_m(\Gamma)(\mathbf{x}_i, \mathbf{x}_j) = |\Gamma|^{-2} \cdot \sum_{k=1}^{|\Gamma|} \left[ C_{ij}^k \cdot (C_{ji}^k)^T + \iota \cdot C_{ij}^k \cdot (C_{ji}^k)^\iota \cdot \iota^{-1} \right]^m.$$ 

0.10. Abstract layout algorithm. The input of the algorithm is the common line data $\{ (C_{ij}, C_{ji}) : i \neq j \}$. The output of the algorithm is a collection of algebra characters $\delta_1, \ldots, \delta_N : A(\Gamma) \to \mathbb{C}$ such that $\delta_i \circ g = \delta_\mathbf{x}_i$ for some element $g \in N^N(\Gamma)/\Gamma$. We notice, that the output of the algorithm establishes a solution to the computational problem as a result of the faithfulness of the relaxation rule $\mathbf{x} \mapsto \delta_\mathbf{x}$.

Remark 0.12. In practice, one needs to solve a non-linear optimization problem in order to reconstruct the orbit $\mathbf{x}$ from its corresponding character $\delta_\mathbf{x}$. However, the number of parameters in this problem is small and does not depend on the number of images.

The core of the algorithm is to compute the fundamental algebra $A(\Gamma)$. In fact, its enough to compute finitely many components since the algebra $A(\Gamma)$ which is isomorphic to the invariant algebra $A(\Gamma)$ is finitely generated. The precise number of components that needs to be computed depends on the subgroup $\Gamma$; in general $A(\Gamma)$ is generated by three homogenous polynomials $P_1, P_2$ and $P_3$, where

$$P_1 \in A_{m_1}(\Gamma),$$
$$P_2 \in A_{m_2}(\Gamma),$$
$$P_3 \in A_{m_3}(\Gamma),$$

satisfying a unique relation. For example, in case $\Gamma$ is the Octahedral subgroup (symmetry of the cube), we have that $m_1 = 4, m_2 = 6$ and $m_3 = 9$ and the relation satisfied by $P_1, P_2$ and $P_3$ is

$$P_2^2 + P_3^3 + P_4^3 P_6 = 0.$$ 

Hence, in this particular case its enough to compute the graded components $A_4(\Gamma)$, $A_6(\Gamma)$ and $A_9(\Gamma)$. The computation of the various graded components uses their spectral characterization as maximal eigenspaces of the transport operators. The
computation uses the approximation scheme in order to compute numerical approximations of the relevant transport operator. For example, in case of the Octahedral subgroup, we only need to compute numerical approximations of the operators $T_4(\Gamma)$, $T_6(\Gamma)$ and $T_9(\Gamma)$.

**Remark 0.13.** We note that, in addition, to the relevant graded components of the fundamental algebra, we require also to numerically compute the defining relation of the algebra and also account for its real structure. These aspects are explained in detail in the body of the paper.

The admissibility of the algorithm amounts to the fact that the map $\tau : A(\Gamma) \to A(\Gamma)$ is an isomorphism of graded real algebras in conjunction with the fact that the automorphism group of the real algebra $A(\Gamma)$ satisfy

$$\text{Aut}_R(A(\Gamma)) = N'(\Gamma)/\Gamma.$$

Finally, the numerical stability of the algorithm amounts to the fact that the transport operators $T_{m_1}(\Gamma), T_{m_2}(\Gamma)$ and $T_{m_3}(\Gamma)$ admit a spectral gap.

0.11. **Structure of the paper.** The remainder of this paper consists of five sections:

- **Section 1** is devoted to the description and study of the non-linear intrinsic reconstitution algorithm in the context of the frame reconstruction problem. We begin by describing the underlying mathematical structures: the fundamental cone and the fundamental algebra. We proceed by describing the representation theoretic and spectral characterizations of the fundamental algebra. In the latter case, we describe the various transport operators and their kernels. The main technical statement is Theorem 1.5 that gives a full description of the spectrum of the various transport operators. This constitutes the main mathematical development of this paper. We end this section by presenting the frame reconstruction algorithm.

- **Section 2** is devoted to the description and study of the non-linear intrinsic reconstitution algorithm in the more general context of the orbit reconstruction problem. We begin by describing the generalizations of the fundamental cone and the fundamental algebra to this more general circumstances. In particular, we give a complete description of the algebra of regular functions on the fundamental cone (Theorem 2.2) and state its remarkable rigidity property (Theorem 2.3) for the various finite subgroups in $SO(V)$. We proceed by describing the spectral characterizations of the fundamental algebra in terms of the various invariant transport operators. In this context, we give a complete description of the spectrum of the various invariant transport operators (Theorem 2.5). We end this section by presenting the orbit reconstruction algorithm.

- **Section 3** is devoted to the proof of theorem 1.5. The fundamental statement is Theorem 3.1, which establish an explicit relation between the transport operators of different levels.

- **Section 4** is devoted to the study of invariant theory with respect to a three Platonic subgroups in $SO(V)$. The main result of this section is description of the complete isotypic decomposition of the homogenous components of the algebra of regular functions on the fundamental cone, with respect to the
various Platonic subgroups. As a consequence, we calculate the dimensions of the corresponding invariant subspaces.

- Appendix A is devoted to the proofs of all the technical statements that appears in the body of the paper.

0.12. **Basic terminology.** The following terminology is used throughout the paper.

0.12.1. **Function spaces.** Given a measured set \((X, \mu)\), we will use the notation \(C(X)\) for denoting any Hilbertian space of complex valued functions on \(X\), where the Hermitian product is taken to be the standard inner product

\[ \langle f, g \rangle = \int_{x \in X} f(x) \overline{g(x)} \mu \]

for every \(f, g \in C(X)\). In particular we don’t explicitly distinguish between an Hilbertian space and its completion; the correct choice depends on the context and is usually obvious, hence, is left to the reader. In the case \(X\) is a smooth manifold, the default convention is the Hilbertian space of square integrable \(C^\infty\) functions. Note that, in case \(X\) is a finite set and \(\mu\) is the counting measure, there is no ambiguity - \(C(X)\) must be the vector space of complex valued functions on \(X\).

0.12.2. **Group actions.** Given a group \(G\) and a manifold \(X\). A left group action \(\triangleright: G \times X \to X\) induces an action on \(C(X)\), given by \((g \cdot f)(x) = f(g^{-1} \triangleright x)\), for every \(f \in C(X)\). A right group action \(\triangleleft: X \times G \to X\) induces an action on \(C(X)\), given by \((g \cdot f)(x) = f(x \triangleleft g)\), for every \(f \in C(X)\).

0.12.3. **Vector spaces, algebras and varieties.** Vector spaces: we only consider vector spaces defined over the field \(\mathbb{C}\) of complex numbers. A real vector space, is a vector space \(V\), equipped with a real structure, which is an anti-complex involution \(\text{conj} : V \to V\). A morphism of real vector spaces is a linear transformation \(T : V_1 \to V_2\) that commutes between the real structures on its domain and range, namely

\[ T \circ \text{conj}_1 = \text{conj}_2 \circ T. \]

Algebras: we only consider algebras defined over \(\mathbb{C}\). A real algebra is an algebra \(A\) equipped with a real structure, i.e., an anti-complex involution \(\text{conj} : A \to A\) that satisfies, in addition,

\[ \text{conj}(a + b) = \text{conj}(a) + \text{conj}(b), \]

\[ \text{conj}(a \cdot b) = \text{conj}(a) \cdot \text{conj}(b), \]

for every \(a, b \in A\). A morphism of real algebras is a morphism of algebras the commutes between the real structures.

Varieties: we only consider affine algebraic varieties defined over \(\mathbb{C}\), that is algebraic varieties whose underlying topological space is the set of algebra characters of some Noetherian algebra. A real variety is an affine algebraic variety associated to a real algebra. Geometrically, a real variety is a variety \(Y\) equipped with an involution

\[ \text{conj} : Y \to Y, \]

however, this picture only shows the topological action of the involution and should be supplemented with the action of the involution on the algebra of regular functions on \(Y\).
Finally, we use the same notation \( \text{conj} \) for denoting the real structure on various different vector spaces and algebras.

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1. **The frame reconstruction problem**

1.1. **Set-up.** The frame manifold is equipped with two commuting actions: a left action of the group \( \text{SO}(V) \), given by composition from the left \( x \mapsto g \circ x \) and a right action of the special orthogonal group \( \text{SO}(3) \), given by composition from the right \( x \mapsto x \circ g \). We denote by \( T_0 \subset \text{SO}(3) \) the subgroup of rotations around the viewing direction, namely, \( T_0 \) is the copy of \( \text{SO}(2) \), consisting of matrices of the form

\[
\begin{pmatrix}
* & * & 0 \\
* & * & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

We distinguish a particular element \( J \) in the normalizer of \( T_0 \), given by

\[
J = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]

The element \( J \) defines a real structure on the algebra \( \mathbb{C}(X) \), given by

\[
\text{conj}(f)(x) = f(x \circ J).
\]

Finally, we identify \( T_0 \) with the circle group \( S^1 \), by sending \( e^{i\theta} \in S^1 \) to the matrix

\[
\begin{pmatrix}
\cos(\theta) & -\sin(\theta) & 0 \\
\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

To summarize, we consider the frame manifold as a principal \( S^1 \) bundle over the unit sphere \( S(V) \), where the fibration map \( \pi : X \to S(V) \) sends a frame \( x \) to its viewing direction \( \pi(x) = e_3 \). Finally, we require the following definition:

**Definition 1.1.** An ordered pair of frames \( (x, y) \in X \times X \) are said to be in **generic position** if their viewing directions satisfy \( \pi(x) \neq \pm \pi(y) \).
1.2. **Fundamental cone.** Our first task is to recast the configuration space in an algebraic setting by replacing the frame manifold with an algebraic variety. Our approach is based on the observation that the unit sphere $S(V)$ admits a structure of a projective algebraic variety, called the Riemann sphere. To make this precise, let $W = \mathbb{C}V$ denote the complexified vector space, equipped with the induced Hermitian product
\[ h(u + iv, u' + iv') = r(u, u') - r(v, v') + ir(v, u') - ir(u, v'), \]
and let us consider the map
\[ X \to W, \]
sending a frame $x$ to the vector $(x) = e_1 - ie_2$. It is easy to verify that the vector $(x)$ satisfies the condition $r((x), (x)) = 0$, which means that the image of the map $\delta$ lies in the cone $Y = \{ w \in W : r(w, w) = 0 \}$, called, the *fundamental cone*. In fact, one can check that the image of $\delta$ coincides with the subset $Y_1 \subset Y$, consisting of points $y \in Y$ that satisfy, in addition, $\frac{1}{2} \cdot h(y, y) = 1$. We consider the cone $Y$ as a real algebraic variety, with the real structure, given by complex conjugation, noting that the set of real points $Y(\mathbb{R})$ is empty. In addition, we observe that the Riemann sphere is the quotient $\mathbb{R} = \mathbb{C}^\times$ and the map $\delta$ induces an isomorphism of manifolds between the unit sphere and the Riemann sphere, establishing the desired projective structure on the unit sphere.

The map $\delta$ is a morphism of $SO(V) \times S^1$ spaces and, moreover, it translates the real structure involution on $Y$ to multiplication from the right by the matrix $J$ on $X$, that is, we have
\[ (1.1) \quad \delta(x \circ J) = \text{conj}(\delta(x)), \]
for every $x \in X$. Let us denote by $\text{Aut}_R(Y)$ the group of automorphisms of the variety $Y$, preserving the real structure.

**Proposition 1.2.** We have
\[ \text{Aut}_R(Y) = SO(V) \times \mathbb{R}^\times. \]
For a proof, see Appendix A.

We observe that one can read the frame $x \in X$ from its corresponding complex vector $\delta(x) \in Y$, therefore, given a projection image $I_i \in \mathcal{P}$, we can trade the reconstruction of the corresponding frame $x_i \in X$ with the "algebraic" problem of reconstructing the point $\delta(x) \in Y$.

1.3. **Fundamental algebra.** Our goal is to exhibit a distinguished real graded algebra $\mathcal{A} \subset \mathbb{C}(X)$, called the *fundamental algebra*, which, as will be explained in the sequel, can be, principally, computed from the projection images and used in order to reconstruct the frame associated with every image. The idea is to realize the algebra of regular functions on $Y$ as a subalgebra of $\mathbb{C}(X)$. We begin by considering the pullback map
\[ \delta^*: \text{Poly}(W) \to \mathbb{C}(X), \]
induced from the map $\delta$. Since $\text{Im} \delta \subset Y$, the pullback map factors through the quotient algebra $\mathcal{A} = \text{Poly}(W)/(r)$, which is the algebra of regular functions on the cone $Y$. The algebra $\mathcal{A}$ is graded and admits a real structure, induced from
complex conjugation. By condition (1.1), the pull-back map becomes a morphism of real algebras. The fundamental algebra is defined as the image

$$\mathcal{A} = \text{Im} \delta^* = \bigoplus_{m=0}^{\infty} \mathcal{A}_m,$$

where $\mathcal{A}_m = \delta^*(A_m)$, for every $m \geq 0$. The fundamental algebra is the core algebraic structure that underlies the reconstruction algorithm, therefore, we will spend considerable space for its study. We proceed by giving it two characterizations: one in terms of representation theory, which is important for obtaining appropriate interpretation of its various graded components and the other in terms of spectral theory, which is the basis for its computation from the set of projection images.

1.4. Representation theory. Equipping $X$ with the unique normalized Haar measure $\mu_{\text{Haar}}$, the resulting Hilbertian space $\mathbb{C}(X)$ is a unitary representation of the group $SO(V) \times S^1$, where the unitary action is induced from the left and right actions of $SO(V)$ and $S^1$ on the manifold $X$ respectively. Consequently, the action of $S^1$ induces an isotypic decomposition

$$\mathbb{C}(X) = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}(X)_m,$$

where the isotypic component $\mathbb{C}(X)_m$ consists of functions $f \in \mathbb{C}(X)$ that satisfy the equivariance condition $f(x \cdot e^{i\theta}) = e^{im\theta} f(x)$, for every $x \in X$ and $\theta \in [0, 2\pi]$. Furthermore, since the actions of $SO(V)$ and $S^1$ commute, each isotypic component in (1.2) is closed under the action of $SO(V)$, thus decomposes further into

$$\mathbb{C}(X)_m = \bigoplus_{n=0}^{\infty} \mathbb{C}(X)_{n,m}.$$

Before we proceed, we remind the reader that irreducible representations of $SO(V)$ are indexed by natural numbers: for $n \in \mathbb{N}$ there exists a unique irreducible representation of dimension $2n + 1$. The following theorem asserts that each isotypic component in (1.3) consists of a single irreducible representation.

**Theorem 1.3** (Multiplicity one). We have

$$\dim \mathbb{C}(X)_{n,m} = \begin{cases} 2n + 1 & n \geq |m| \\ 0 & n < |m| \end{cases}.$$

For a proof, see Appendix A. The proof is principally an application of the Peter Weyl theorem.

The main conclusion is that the fundamental algebra can be characterized in terms of the isotypic decomposition (1.3).

**Theorem 1.4.** We have

$$\mathcal{A}_m = \mathbb{C}(X)_{m,-m},$$

for every $m \geq 0$.

For a proof, see Appendix A.
1.5. Spectral theory. The various graded components of the fundamental algebra can be characterized as the maximal eigenspaces of various integral operators acting on $C(X)$. In more precise terms, the $m$th graded component turns out to be the maximal eigenspace of a particular integral operator $T_m$, $m \geq 0$, called the transport operator of level $m$. We proceed to describe these operators, by specifying their kernel functions, noting that the integration is taken with respect to the Haar measure $\mu_{\text{Haar}}$. We will often confuse between the integral operator and its kernel, in particular, we use the same notation for both of them.

1.5.1. Common line matrices. The kernels of the various transport operators are given by complex valued distributions $T_m \in D'(X \times X)$, that are smooth on the open set, consisting of pairs of frames in generic position (see Definition 1.1). For such pairs, the value of the various transport kernels can be derived from a more fundamental $2 \times 2$ real matrix, called the common line matrix, defined as follows: given a pair of frames $(x, y) \in X \times X$ in generic position, the common line matrix associated to this pair, is defined as

$$C(x, y) = p(x)^T \circ p_L \circ p(y),$$

where, $p_L : V \rightarrow V$ is the orthogonal projection on the line of intersection (common line) $L = \text{Im} p(x) \cap \text{Im} p(y)$. Note that we are using the fact that $(x, y)$ are in generic position in order to ensure that the common line is non-degenerate. In plain language, the common line matrix identifies the two coordinate realizations of the common line, obtained via the maps $p(x)$ and $p(y)$.

1.5.2. Transport kernels. We begin by specifying the kernel of the transport operator of level 1. To this end, we identify the real plane $\mathbb{R}^2$ with the field of complex numbers $\mathbb{C}$ and average the common line matrix with respect to the complex multiplication, obtaining the $1 \times 1$ complex matrix

$$(1.4) \quad T_1(x, y) = C(x, y) + i \cdot C(x, y) \cdot i^{-1},$$

for every pair of frames $(x, y)$ in generic position, where $i = \sqrt{-1}$. It is not difficult to verify that, in fact, $T_1(x, y) \in S^1$. The kernel of the transport operator of level $m$, is defined as

$$T_m(x, y) = T_1(x, y)^m,$$

for every $m \geq 0$ and $(x, y)$ in generic position. Although, apriori specified only for pairs of frames in generic position, a closer inspection reveals that the various transport kernels are in fact smooth on the bigger open set, consisting of all pairs $(x, y)$ that satisfy only the condition $\pi(x) \neq -\pi(y)$. The following properties of the transport kernels can be easily verified:

- Symmetry property: $T_m(x, y) = T_m(y, x)$, for every pair of frames $(x, y)$ such that $\pi(x) \neq -\pi(y)$.
- Invariance property: $T_m(g \circ x, g \circ y) = T_m(x, y)$, for every pair of frames $(x, y)$ such that $\pi(x) \neq -\pi(y)$ and $g \in SO(V)$.
- Equivariance property: $T_m(x \circ e^{i\theta_1}, y \circ e^{i\theta_2}) = e^{im(\theta_2 - \theta_1)}T_m(x, y)$, for every pair of frames $(x, y)$ such that $\pi(x) \neq -\pi(y)$ and angles $\theta_1, \theta_2 \in [0, 2\pi]$.
- Reality property: $T_m(x, y \circ J) = T_m(x \circ J, y)$, for every pair of frames $(x, y)$ such that $\pi(x) \neq -\pi(y)$. 

Note that the symmetry property implies that the transport operators are (formally) self-adjoint, the invariance property implies that the transport operators are intertwiners with respect to the $SO(V)$-action and the reality property implies that the transport operators commute with the real structure \( ? ? \). The implication of the equivariance property will be explained below when we discuss the spectral decomposition of the transport operators.

1.5.3. Spectral properties of the transport operators. The spectral decomposition of the various transport operators can be explained in terms of the isotypic decomposition (1.3). The following theorem consists of the main technical statement of this paper:

**Theorem 1.5** (Spectral theorem). We have

1. The isotypic component \( C(X)_{n,l} \subset \ker T_m \), for every \( l \neq -m \).
2. The operator \( T_m \) acts on the isotypic component \( C(X)_{n,-m} \), as a scalar operator \( \lambda_{n,m} \cdot \text{Id} \). Moreover

\[
\lambda_{n,m} = \frac{(-1)^{n-m} m}{n(n+1)},
\]

for every \( n \geq m \geq 1 \).

For a proof, see Section 3.

Theorem 1.5 has two important consequences: first, in conjunction with Theorem 1.4, it implies that the \( m \)th graded component of the fundamental algebra is the maximal eigenspace of the transport operator \( T_m \).

**Theorem 1.6.** We have

\[ A_m = \text{maximal eigenspace of } T_m, \]

for every \( m \geq 1 \).

The second consequence, is that there exists a spectral gap that separates the maximal eigenvalue of \( T_m \) from the rest of the spectrum, which is summarized in the following theorem:

**Theorem 1.7** (Spectral gap). There exists a spectral gap \( \text{gap}_m > 0 \) that separates the maximal eigenvalue of \( T_m \) from the rest of the spectrum. Moreover,

\[
\text{gap}_m = \lambda_{m,-m} - \lambda_{m+2,-m} = \frac{4m+6}{(m+1)(m+2)(m+3)},
\]

for every \( m \geq 1 \).

Note that \( \text{gap}_1 = 5/12 \) and \( \text{gap}_m \) decreases as \( m \) increases.

1.6. Algorithm. For the sake of clarity we describe the algorithm in several stages: first stage, we describe its output and explain the sense by which it establish a solution to the frame reconstruction problem. Second stage, we describe a simplified layout of the algorithm, disregarding various numerical aspects. Finally, in the third stage, we describe the actual algorithm, taking into account all the numerical aspects.
1.6.1. Output of the algorithm. The goal of the computation is to obtain an estimate of the frame map, which is reduced to obtaining an estimate of the "algebraic" map \( \Phi = \delta \circ \text{Fr} \), since we know that the frame map can be reconstructed from this composition. The optimization paradigm that we are about to present, characterizes the solution for the reconstruction problem, as the pair

\[
\text{Sol} = (Y, \Phi),
\]

considered as an object in some appropriately defined category (we will not use categorical language in this paper any further). However, the output of the algorithm is another object

\[
\hat{\text{Sol}} = (\hat{Y}, \hat{\Phi}),
\]

where \( \hat{Y} \) is real cone, isomorphic to \( Y \) and \( \hat{\Phi} : \mathcal{P} \rightarrow \hat{Y} \). That object (1.6) is isomorphic (up-to some distortion) to the "true" object given in (1.5) and should be considered as an estimator for the "true" object, in the sense that there exists an isomorphism of real cones \( \tau : \hat{Y} \rightarrow Y \), such that

\[
\tau \circ \hat{\Phi} \sim \Phi.
\]

Let us now explain what can be derived from this kind of output. In practice, the isomorphism \( \tau \) is not known, however, by choosing an arbitrary isomorphism of real cones \( \tau' : \hat{Y} \rightarrow Y \), there exist an element \( g \in \text{Aut}_R(Y) = SO(V) \times \mathbb{R}^\times \), such that

\[
\tau' \circ \hat{\Phi} \sim g \cdot \Phi.
\]

Furthermore, by imposing the condition \( \text{Im} \Phi \subset Y_1 \), the scaling factor can be reduced to an element of the subgroup \( \mu_2 = \{1, -1\} \). Thus, concluding that we are able to reconstruct the true map \( \Phi \) up-to an action of a fixed element in the full orthogonal group \( O(V) = SO(V) \times \mu_2 \), which, in turns, enables us to reconstruct the potential function \( \phi \) up to a fixed rotation in \( O(V) \).

Next, we have to explain, what does it mean to compute a pair consisting of an algebraic variety and a finite set of points inside it. In principle, what we need to do, is to express the output of the computation in the language of linear algebra. To this end, we invoke the fundamental duality of algebraic geometry, relating algebras with geometric spaces (affine algebraic varieties), or more precisely, relating an algebra with the space, whose points consists of all algebra characters. To summarize:

- The computation of the algebraic variety \( \hat{Y} \) amounts to the computation of a real graded algebra \( \hat{A} \).
- The computation of the map \( \hat{\Phi} \) amounts to the computation of an algebra character \( \hat{\Phi}(I) : \hat{A} \rightarrow \mathbb{C} \), for every image \( I \in \mathcal{P} \).
- The isomorphism \( \tau \) amounts to an isomorphism of real graded algebras \( \tau : A \rightarrow \hat{A} \), satisfying

\[
\hat{\Phi}(I) \circ \tau \sim \Phi(I),
\]

for every \( I \in \mathcal{P} \).

1.6.2. Simplified layout of the algorithm. For the sake of clarity, we begin by first describing a simplified layout of the algorithm, in which the approximation relation (1.7) is replaced by the exact equality

\[
\hat{\Phi}(I) \circ \tau = \Phi(I),
\]
for every \( I \in \mathcal{P} \). As we mentioned before, the computation is based on reconstructing the fundamental algebra out of the set of projection images. Since our goal is to reconstruct the fundamental algebra as a real graded algebra we require to introduce a small adjustment to the set-up, in order to account for the real structure on the finite level. We assume that the set of projection images consists, together with every image \( I \), its reflection \( I^r \), given by
\[
I^r (p, q) = I (p, -q),
\]
for every \((p, q) \in \mathbb{R}^2\). This assumption can be easily forced in practice, by extending the initial set of images by adding to every image its reflection with respect to the \( y \)-axis. Granting this assumption, we can define a real structure on the finite Hilbert space \( \mathbb{C} (\mathcal{P}) \), which sends a function \( f \) to the function \( \text{conj} (f) \), defined by
\[
\text{conj} (f) (I) = \overline{f (I^r)},
\]
for every \( I \in \mathcal{P} \). In addition, since we are allowed to assume that the frame map satisfies
\[
\text{Fr} (I^r) = \text{Fr} (I) \cdot J,
\]
for every \( I \in \mathcal{P} \), the pull-back map \( \text{Fr}^* : \mathbb{C} (X) \to \mathbb{C} (\mathcal{P}) \) becomes a morphism of real algebras. The simplified algorithm proceeds in three steps:

1. Compute the subspace
   \[
   \hat{A}_1 = \text{Fr}^* \mathcal{A}_1,
   \]
   the restriction of the first homogenous component of the fundamental algebra to \( \mathcal{P} \).

2. Let \( Q = \ker (\text{Mul}) \), where \( \text{Mul} = \text{Mul}_{\hat{A}_1} : \text{Sym}_2 (\hat{A}_1) \to \mathbb{C} (\mathcal{P}) \) is the map induced from the multiplication on \( \mathbb{C} (\mathcal{P}) \). Define the algebra
   \[
   \hat{A} = \text{Sym}_* (\hat{A}_1) / (Q),
   \]
   the quotient of the symmetric algebra generated by \( \hat{A}_1 \) by the ideal generated by \( Q \).

3. For every image \( I \in \mathcal{P} \), define \( \hat{\Phi} (I) : \hat{A} \to \mathbb{C} \) to be the unique algebra character that satisfies
   \[
   \hat{\Phi} (I) (a) = a (I),
   \]
   for every \( a \in \hat{A}_1 \).

Let us make a few simple observations: first observation is that the algebra \( \hat{A} \) is graded, due to the fact that the vector space \( Q \) which generates the ideal of relations is homogenous, concentrated at degree 2. Second observation is that the algebra \( \hat{A} \) admits a real structure, since the vector spaces \( \hat{A}_1 \) and \( Q \) admit real structures: the vector space \( \hat{A}_1 \) is real, since the morphism
\[
\text{Fr}^* : \mathcal{A}_1 \to \text{Fr}^* \mathcal{A}_1,
\]
is a morphism of real vector spaces and the vector space \( Q \) is real since the multiplication morphism
\[
\text{Mul} : \text{Sym}_2 (\hat{A}_1) \to \mathbb{C} (\mathcal{P}),
\]
is a morphism of real vector spaces. In addition, we have to justify the existence of the algebra character extension in step 3. For this, let us denote by \( \text{ev}_I : \hat{A}_1 \to \mathbb{C} \) the linear functional of evaluation at \( I \) and notice that \( \text{ev}_I \) extends to a linear
functional on $\text{Sym}_2(\hat{A}_1)$ that vanishes on $Q$. Hence, it admits a unique extension to an algebra character of $\hat{A}$.

The admissibility (correctness) of the algorithm is the content of the following theorem:

**Proposition 1.8.** There exists a canonical isomorphism of real graded algebras

$$\tau : A \to \hat{A},$$

satisfying $\hat{\Phi}(I) \circ \tau = \Phi(I)$, for every $I \in \mathcal{P}$.

For a proof see Appendix A.

1.6.3. **Spectral approximation.** In reality, the vector space $\text{Fr}^* \mathcal{A}_1$ can be computed only approximately and the precision of the reconstruction strongly depends on how good is this approximation. The idea is to use the spectral characterization of the fundamental algebra. To this end, let us denote by $T_1$ the restriction of the transport kernel of level 1 to $\mathcal{P} \times \mathcal{P}$, via the frame map, that is

$$T_1 = (\text{Fr} \times \text{Fr})^* T_1. \tag{1.11}$$

**Main observation:** the crucial fact, is that although we have no knowledge about the theoretical vector space $\text{Fr}^* \mathcal{A}_1$, we do know how to compute the finite kernel $T_1$, in the sense that, given a pair of projection images $I, I' \in \mathcal{P}$, the complex scalar $T_1(I, I')$ can be computed out of $I$ and $I'$. We explain this observation in ???.

The corresponding operator $T_1 : \mathbb{C}(\mathcal{P}) \to \mathbb{C}(\mathcal{P})$ is self-adjoint due to the symmetry property of the transport kernel, thus admitting a spectral decomposition

$$\mathbb{C}(\mathcal{P}) = \bigoplus_\lambda \mathbb{C}(\mathcal{P})_\lambda.$$

Since we assume that the frames in $\text{Im} \text{Fr} \subset X$ are distributed independently and uniformly at random, the finite kernel $T_1$ approximates the continuous kernel $T_1$, hence, the vector space $\text{Fr}^* \mathcal{A}_1$ is approximated by the vector space

$$\hat{A}_1 = "\text{maximum } 3\text{D eigenspace}" \text{ of } T_1 = \bigoplus_{\lambda > \lambda_{1,1}} \mathbb{C}(\mathcal{P})_\lambda, \tag{1.12}$$

representing the portion of the spectrum that corresponds to the maximum eigenspace (of multiplicity 3) of the operator $T_1$. Furthermore, the vector space $\hat{A}_1$ admits a real structure, since $T_1$ commutes with the real structure on $\mathbb{C}(\mathcal{P})$, which, in turns, follows from (1.9) in conjunction with the reality property of the continuous kernel $T_1$. The computation now proceeds along the same lines as before, incorporating some small twists.

The algebra $\hat{A}$ is defined to be

$$\hat{A} = \text{Sym}_*(\hat{A}_1)/(\hat{Q}),$$

where $\hat{Q}$ is now taken to be the "effective" kernel of the multiplication map $\text{Mul} = \text{Mul}_{\hat{A}_1} : \text{Sym}_2(\hat{A}_1) \to \mathbb{C}(\mathcal{P})$, that is

$$\hat{Q} = \text{minimum } 1\text{D eigenspace of } \text{Mul} \circ \text{Mul}^*.$$
The algebra character associated to an image $I \in \mathcal{P}$ is taken to be the algebra character whose restriction to $\hat{A}_1$ is the closest to the evaluation functional $ev_I : \hat{A}_1 \to \mathbb{C}$, that is

$$\hat{\Phi}(I) = \min_{ch \in \text{mspec}(\hat{A})} \| ev_I - ch_{|\hat{A}_1} \|,$$

where $\| - \|$ can be taken to be any reasonable norm, for example the one induced from the standard Hilbert structure on $\mathbb{C}(\mathcal{P})$.

2. The orbit reconstruction problem

In this section we explain the generalization of the paradigm, developed in the previous section, to molecules with symmetries. The exposition will be relatively sketchy, as our goal is concerned more with laying down the mathematical foundations than with giving a detailed presentation and analysis of a "working" algorithm. Such a presentation will appear in a future publication.

2.1. Set-up. Given a finite subgroup $\Gamma \subset SO(V)$, we denote by $W(\Gamma)$ the quotient group $N(\Gamma)/\Gamma$ where $N(\Gamma) \subset SO(V)$ is the normalizer of $\Gamma$. We notice that, in the case $\Gamma$ is the trivial subgroup, we have $W(\Gamma) = SO(V)$.

The classification of finite subgroups in $SO(V)$ is well known. There are finite number of possibilities: the trivial group $\Gamma = \{e\}$, two infinite families and three sporadic types corresponding to the symmetries of the Platonic solids.

- Cyclic groups: $\Gamma \simeq C_l$, $l \geq 2$, where $C_l$ is the cyclic group consisting of $l$ elements.
- Dihedral groups: $\Gamma \simeq D_l$, $l \geq 2$, where $D_l$ is the dihedral group consisting of $2l$ elements.

**Platonic solids**

- Tetrahedral group: $\Gamma \simeq A_4$, where $A_4$ is the alternating group of four symbols - this group appears, in particular, as the symmetry group of the standard tetrahedron.
- Octahedral group: $\Gamma \simeq S_4$, where $S_4$ is the permutation group of four symbols - this group appears as the symmetry group of the standard cube.
- Icosahedral group: $\Gamma \simeq A_5$, where $A_5$ is the alternating group of five symbols - this group appears as the symmetry group of the standard icosahedron.

Finally, for the specification of the invariant counterpart of the transport operators, we require the following definition:

**Definition 2.1.** A pair of orbits $\pi, \eta \in \mathbb{X}$ are said to be in generic position if every pair of representatives $x' \in \pi$ and $y' \in \eta$, satisfy $\pi(x') \neq \pm \pi(y')$.

2.2. Fundamental cone. For the reminder of this section, we fix a finite subgroup $\Gamma \subset SO(V)$. As before, our first goal is to recast the orbit map in an algebraic setting by replacing its range with an appropriate algebraic variety. This is achieved by introducing the fundamental cone, which is now given by

$$\mathcal{V} = \Gamma \backslash Y.$$
The algebra of regular functions on $\mathcal{Y}$ is the algebra of invariants
\[(2.2) \quad \mathcal{A} = A^\Gamma.\]

The morphism $\delta$ induces a morphism between quotient spaces
\[(2.3) \quad \overline{\delta} : \mathcal{X} \to \mathcal{Y}.\]

In fact, $\text{Im} \overline{\delta} = \mathcal{Y}_1$, where $\mathcal{Y}_1$ consists of orbits $\overline{y} \in \mathcal{Y}$ such that $1/2 \cdot h(y, y) = 1$, noting that the last condition does not depend on the choice of representative.

Using the fundamental cone, we replace the problem of computing the orbit map with the problem of computing the "algebraic" map
\[
\overline{\mathcal{O}} = \overline{\delta} \circ \mathcal{O},
\]
noting that $\mathcal{O}$ can be reconstructed from $\overline{\mathcal{O}}$. We record two properties of the fundamental cone. First property concerns the structure of its algebra of regular functions. The content of the following theorem amounts to the solution of the first and second fundamental theorems of invariant theory in the context of finite subgroups in $SO(V)$, claiming that for every finite subgroup, the algebra of invariants is generated by three real polynomials $P_{m_i} \in A_{m_i}$, $i = 1, 2, 3$, satisfying a unique relation.

**Theorem 2.2 (Structure Theorem).** The algebra $\mathcal{A}$ admits the following presentation:

1. Cyclic groups $\Gamma \simeq C_l$, $l \geq 2$:
   \[
   \mathcal{A} = \mathbb{C}[P_1, P_l, P_1^l] / (P_1^2 + P_1^{2l} - 2(-1)^l P_1^{2l}),
   \]
2. Dihedral groups $\Gamma \simeq D_l$, $l \geq 2$:
   \[
   \mathcal{A} = \mathbb{C}[P_2, P_1, P_{l+1}] / (P_2 P_1^2 - P_{l+1}^2 - 2(-1)^l P_{l+1}^{2l+1}),
   \]
3. Tetrahedral group $\Gamma \simeq A_4$:
   \[
   \mathcal{A} = \mathbb{C}[P_3, P_4, P_6] / (aP_3^4 + bP_4^3 + cP_6^2),
   \]
   for some $a, b, c$ non-zero real numbers.
4. Octahedral group $\Gamma \simeq S_4$:
   \[
   \mathcal{A} = \mathbb{C}[P_4, P_6, P_3] / (aP_4^3 + bP_6^3 + cP_4^3P_6),
   \]
   for some $a, b, c$ non-zero real numbers.
5. Icosahedral group $\Gamma \simeq A_5$:
   \[
   \mathcal{A} = \mathbb{C}[P_6, P_{10}, P_{15}] / (aP_{10}^2 + bP_{10}^3 + cP_6^5),
   \]
   for some $a, b, c$ non-zero real numbers.

For a proof, see Appendix A. The verification for the Platonic subgroups uses the results and terminology of Section 4.

Second property of the fundamental cone, is a generalization of the rigidity property (see Proposition 1.2), stating that $\mathcal{Y}$ admits no automorphisms except the obvious ones.

**Theorem 2.3 (Rigidity property).** We have
\[
\text{Aut}_\mathbb{R}(\mathcal{Y}) = W(\Gamma) \times \mathbb{R}^\times,
\]
\[
\text{Aut}_\mathbb{R}(\mathcal{Y}_1) = W(\Gamma) \times \mu_2.
\]
For a proof, see Appendix A.

Finally, knowing the generators and relations of the algebra of invariants, we can calculate the dimension of its various homogeneous components. The results are summarized in the following proposition.

**Proposition 2.4.** We have

1. **Cyclic groups** $\Gamma \simeq C_l$, $l \geq 2$:
   \[
   \dim \overline{A}_m = 2n + 1,
   \]
   where $m = nl + r$, $0 \leq r < l$.
2. **Dihedral group** $\Gamma \simeq D_l$, $l \geq 2$:
   \[
   \dim \overline{A}_m = \begin{cases} 
   n + 1 & \text{if } m \text{ even} \\
   n & \text{if } m \text{ odd}
   \end{cases},
   \]
   where $m = nl + r$, $0 \leq r < l$.
3. **Tetrahedral group** $\Gamma \simeq A_4$:
   \[
   \dim \overline{A}_m = \begin{cases} 
   n + 1 & r = 0, 3, 4 \\
   n & \text{otherwise}
   \end{cases},
   \]
   where $m = 6n + r$, $0 \leq r < 6$.
4. **Octahedral group** $\Gamma \simeq S_4$:
   \[
   \dim \overline{A}_m = \begin{cases} 
   n + 1 & r = 0, 4, 6, 8, 9, 10 \\
   n & \text{otherwise}
   \end{cases},
   \]
   where $m = 12n + r$, $0 \leq r < 12$.
5. **Icosahedral group** $\Gamma \simeq A_5$:
   \[
   \dim \overline{A}_m = \begin{cases} 
   n + 1 & r = 0, 6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28 \\
   n & \text{otherwise}
   \end{cases},
   \]
   where $m = 30n + r$, $0 \leq r < 30$.

**2.3. Fundamental algebra.** The main underlying structure behind the orbit reconstruction algorithm is a distinguished real graded subalgebra $\mathcal{A} \subset \mathbb{C}(X)$, called the fundamental algebra. The fundamental algebra is defined to be

\[
\mathcal{A} = \Im \delta^*,
\]

where, $\delta^* : \overline{A} \to \mathbb{C}(X)$ is the pull-back map. The upshot is that the fundamental algebra can be computed from the projection images and used in order to compute the orbit associated with every image. The computation relies on the fact that the fundamental algebra admits a spectral characterization, which identifies the $m$'th homogeneous component with the maximal eigenspace of a specific integral operator $\mathcal{T}_m$ acting on $\mathbb{C}(X)$, called the *invariant transport operator* of level $m$.

We describe these operators by specifying their kernel, noting that the integration is taken with respect to the "Haar" measure $\bar{\mu}_{\text{Haar}}$ on $X$ - the push-forward of the unique normalized Haar measure on $X$, via the canonical projection map.
2.3.1. The kernels of the invariant transport operators. The kernel of the invariant transport operator of level $m$ is given by a distribution $\mathcal{T}_m \in D'(X \times X)$, smooth on the open set of pairs of orbits in generic position. Given such a pair $(\pi, \gamma)$, in order to specify the value of $\mathcal{T}_m$, we consider the vector spaces $\mathbb{C}(\pi)$ and $\mathbb{C}(\gamma)$, consisting of complex valued functions on the orbits $\pi$ and $\gamma$ respectively and introduce the auxiliary operator $\mathcal{T}_m(\pi, \gamma): \mathbb{C}(\gamma) \to \mathbb{C}(\pi)$, given by

$$
\mathcal{T}_m(\pi, \gamma)(f)(x') = \frac{1}{|\gamma'|} \sum_{y' \in \gamma} \mathcal{T}_m(x', y') f(y'),
$$

for every $f \in \mathbb{C}(\gamma)$. The invariance property of the transport kernel implies that $\mathcal{T}_m(\pi, \gamma)$ is an intertwiner with respect to the $\Gamma$-action on its domain and range, hence it restricts to an operator between the subspaces of $\Gamma$-invariant vectors. In addition,

$$
\mathbb{C}(\gamma)^\Gamma = \mathbb{C}(\pi)^\Gamma = \mathbb{C},
$$

since the group $\Gamma$ acts transitively on the corresponding orbits, hence the only invariant functions are the constant functions. The value of the kernel of $\mathcal{T}_m$ at $(\pi, \gamma)$ is defined by the condition

$$
\mathcal{T}_m(\pi, \gamma) \neq 0.
$$

2.3.2. The spectral decomposition of the invariant transport operators. In order to describe the spectrum of the invariant transport operators, we introduce a decomposition that is induced from the isotypic decomposition (1.3). To this end, we identify $\mathbb{C}(X) = \mathbb{C}(X)^\Gamma$ and consider the decomposition

$$(2.4) \quad \mathbb{C}(X) = \bigoplus_{m \in \mathbb{Z}} \bigoplus_{n \geq |m|} \mathbb{C}(X)_{n, m},$$

where $\mathbb{C}(X)_{n, m} = \mathbb{C}(X)^\Gamma_{n, m}$. The spectral decomposition of the operator $\mathcal{T}_m$ can be interpreted in terms of the "isotypic" decomposition (2.4). The following theorem is a generalization of the spectral theorem 1.5.

**Theorem 2.5.** We have

1. The "isotypic" component $\mathbb{C}(X)_{n, l} \subset \ker \mathcal{T}_m$, for every $l \neq -m$.
2. The operator $\mathcal{T}_m$ acts on the "isotypic" component $\mathbb{C}(X)_{n, -m}$, as a scalar operator $\lambda_{n, m} \cdot \text{Id}$, where

$$
\lambda_{n, m} = (-1)^{n-m} \frac{m}{n(n+1)},
$$

for every $n \geq m \geq 1$ such that $\mathbb{C}(X)_{n, -m} \neq 0$.

For a proof, see Appendix A.

The main consequence of Theorem 2.5 is that $\mathcal{A}_m$ coincides with the maximal eigenspace of the invariant transport operator of level $m$, for every $m \geq 0$ and, moreover, there exists a spectral gap that separates this eigenspace from the rest of the spectrum, which is crucial for ensuring the numerical stability of the reconstruction algorithm. This is summarized in the following theorem.

**Theorem 2.6.** If $\mathcal{A}_m \neq 0$ then

$$
\mathcal{A}_m = \text{maximum eigenspace of } \mathcal{T}_m,
$$
and, moreover, there exists a spectral gap that separates $\mathcal{A}_m$ from the rest of the spectrum of $\mathcal{T}_m$.

The spectral gap can be evaluated precisely, for every finite subgroup $\Gamma$ and homogenous degree $m$, using the explicit dimension values, specified in Proposition 2.4.

2.4. Algorithm. The orbit reconstruction algorithm is based on the same principles as the frame reconstruction algorithm, described in the previous section. However, in these more general circumstances, the utilization of the full structure of the fundamental algebra becomes more apparent.

2.4.1. Output of the algorithm. The goal of the computation is to obtain an estimate of the orbit map, which is reduced to obtaining an estimate of the "algebraic" map $\Phi = \delta \circ \text{Or}$. The starting point is to characterize the solution of the reconstruction problem as a pair

$$\text{Sol} = (\mathcal{Y}, \Phi),$$

considered as an object in some appropriately defined category. The output of the algorithm is another object

$$\hat{\text{Sol}} = (\hat{\mathcal{Y}}, \hat{\Phi}),$$

where $\hat{\mathcal{Y}}$ is real cone, isomorphic to $\mathcal{Y}$ and $\hat{\Phi} : \mathcal{P} \rightarrow \hat{\mathcal{Y}}$. The object (2.6) is isomorphic (up-to some distortion) to the "true" object given in (2.5), or, more precisely, should be considered as an estimator, in the sense that there exists an isomorphism of real cones $\tau : \hat{\mathcal{Y}} \rightarrow \mathcal{Y}$, such that

$$\tau \circ \hat{\Phi} \sim \Phi.$$

This implies that the estimation of the "true" object can be derived up-to an action of the automorphism group of the fundamental cone. This is because, in practice, the map $\tau$ is not known, hence by choosing an arbitrary isomorphism of real cones $\tau' : \hat{\mathcal{Y}} \rightarrow \mathcal{Y}$, we only have that

$$\tau' \circ \hat{\Phi} \sim g \cdot \Phi,$$

for some element $g \in \text{Aut}_\mathbb{R} (\mathcal{Y}) = W (\Gamma) \times \mathbb{R}^\times$. By imposing the condition $\text{Im} \hat{\Phi} \subset \mathcal{Y}_1$, the scaling factor can be reduced to an element of the subgroup $\mu_2 = \{1, -1\}$.

Thus, concluding that we are able to reconstruct the true map $\Phi$ up-to an action of a fixed element in the normalizer of $\Gamma$ in the full orthogonal group $O (V)$, a group which is equal to $N (\Gamma) \times \mu_2$. This enables us to reconstruct the potential function $\phi$ up to a fixed rotation in $N (\Gamma) \times \mu_2$.

2.4.2. Simplified layout of the algorithm. We begin by describing a simplified layout of the algorithm, in which, the approximated relation (2.7) is replaced by exact equality:

$$\tau \circ \hat{\Phi} = \Phi.$$

Based on this layout we proceed to describe the actual algorithm. For the sake of clarity we assume that the symmetry group $\Gamma$ is isomorphic to $S^4$ - the octahedral symmetry. In this particular case, the algebra of invariants is generated by three generators, laying in degrees $m_1 = 4, m_2 = 6$ and $m_3 = 9$ and, moreover, the corresponding homogenous components are one dimensional (see Proposition 2.4), a property that simplifies the presentation a bit. In addition, we recall the assumption
that the set of images includes together with every image its reflection with respect to the $y$-axis, an assumption that allows us to account, on the finite level, for the real structure of the fundamental algebra.

The algorithm proceeds in the three steps:

1. Compute the subspaces
   \[ \hat{A}_{m_1} = \text{Fr}^*A_{m_1}, \]
   \[ \hat{A}_{m_2} = \text{Fr}^*A_{m_2}, \]
   \[ \hat{A}_{m_3} = \text{Fr}^*A_{m_3}. \]

2. Let $Q = \ker(m)$, where $m$ is the map
   \[ m : \text{Sym}_3(\hat{A}_{m_2}) \oplus \text{Sym}_2(\hat{A}_{m_3}) \oplus \text{Sym}_3(\hat{A}_{m_1}) \otimes \hat{A}_{m_2} \rightarrow \mathbb{C}(\mathcal{P}), \]
   induced from the multiplication map on $\mathbb{C}(\mathcal{P})$. Define the algebra
   \[ \hat{A} = \text{Sym}_* ( \hat{A}_{m_1} \oplus \hat{A}_{m_2} \oplus \hat{A}_{m_3} )/ (Q), \]

3. For every image $I \in \mathcal{P}$, define $\hat{\Phi}(I) : \hat{A} \rightarrow \mathbb{C}$ to be the unique algebra character that satisfies
   \[ \hat{\Phi}(I)(a) = a(I), \]
   for every $a \in \hat{A}_{m_1} \oplus \hat{A}_{m_2} \oplus \hat{A}_{m_3}$.

In step 2, we choose the domain of the map $m$, based on the precise form of the relation satisfied by the generators of the algebra of invariants in the case of octahedral symmetries (see Theorem 2.2). Moreover, notice that, for this particular group, the algebra of invariants is generated by the vector space $\mathcal{A}_{m_1} \oplus \mathcal{A}_{m_2} \oplus \mathcal{A}_{m_3}$ and the ideal is generated by a unique polynomial in degree $m_3^3 = m_2^2 = m_1$ justifying the definition of the algebra $\hat{A}$. Indeed, it is not difficult to verify that there exists an isomorphism of real graded algebras $\tau : \mathcal{A} \rightarrow \hat{A}$, satisfying
   \[ \hat{\Phi}(I) \circ \tau = \overline{\Phi}(I), \]
   for every $I \in \mathcal{P}$ (see the proof of Proposition 1.8).

2.4.3. Spectral approximation. In reality, the subspaces $\text{Fr}^*A_{m_i}$, $i = 1, 2, 3$, can be computed only approximately and the precision of the reconstruction strongly depends on how good are these approximations. An approximation can be obtained using the spectral characterization of the various homogenous components of the fundamental algebra. The development is principally identical to what we did in the previous section, hence, we discard most of the details.

The spectral approximation of the theoretical subspaces $\text{Fr}^*A_{m_i}$, $i = 1, 2, 3$, are given by the vector spaces
   \[ \hat{A}_{m_i} = \text{maximum 1D eigenspace of } \overline{T}_{m_i}, \]
for $i = 1, 2, 3$, where $\overline{T}_{m_i}$ is the restriction of the invariant transport kernel $\overline{T}_{m_i}$, via the orbit map. The main point is that the finite kernel can be computed out of the projection images, in the sense that the value $\overline{T}_{m_i}(I, I')$ can be computed from the images $I$ and $I'$. The computation is an elaboration of the computation of the single transport matrix that we explained in the previous section. Similarly, it is based on the derivation of the transport matrix from the common line matrix combined with the fact the the common line matrix can be computed from the projection images.
2.5. **Computation of the symmetry group.** We conclude this section with the explanation of how to compute the symmetry group out of the projection images. There are several ways to do that. One is to read the symmetry from the spectrum of the invariant transport operator $T_1$, or, more precisely, from its finite approximation $T_1$. Let us write eigenvalues of $T_1$ in decreasing order according to their absolute value $|\lambda_1| \geq |\lambda_2| \geq .. \geq |\lambda_n| \geq .. \geq 0$ and denote the multiplicity of $\lambda_n$ by $m_n$. The following table lists the pairs $(\lambda_1, m_1)$ for the different subgroups $\Gamma \subset SO(V)$.

<table>
<thead>
<tr>
<th>Group</th>
<th>$(\lambda_1, m_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trivial:</td>
<td>${ e }$</td>
</tr>
<tr>
<td>Cyclic:</td>
<td>$C_l, l \geq 2$</td>
</tr>
<tr>
<td>Dihedral:</td>
<td>$D_l, l \geq 3$</td>
</tr>
<tr>
<td>Tetrahedral:</td>
<td>$A_4$</td>
</tr>
<tr>
<td>Octahedral:</td>
<td>$S_4$</td>
</tr>
<tr>
<td>Icosahedral:</td>
<td>$A_5$</td>
</tr>
</tbody>
</table>

From the table, it can be seen, that the pair $(\lambda_1, m_1)$ yields a spectral signature, distinguishing between the different group types. Furthermore, in case the subgroup $\Gamma$ belongs to one of the infinite families of cyclic or dihedral subgroups, the degree of $\Gamma$ can be determined from the spectrum of $T_1$ as follows:

- In case $\Gamma \simeq C_l, l \geq 2$, we have $(\lambda_1, m_1) = ((-1)^{l-1} / l (l + 1), 3)$, while $m_n = 1$, for every $n < l$. Thus the degree $l$ can be read from the first eigenvalue of multiplicity $3$.
- In case $\Gamma \simeq D_l, l \geq 2$, let $n_0$ denote the first index such that $m_{n_0} = 2$. If $l$ is even then $\lambda_{n_0} = (-1)^{l-1} / l (l + 1)$ and $\lambda_{n_0-1} \leq 0$. If $l$ is odd then $\lambda_{n_0} = (-1)^{l} / (l + 1) (l + 2)$ and $\lambda_{n_0-1} > 0$. Thus, the degree $l$ can be read from the first eigenvalue of multiplicity $2$ in conjunction with the sign of the previous eigenvalue.

3. **Spectral theory of the transport operators**

In this section we prove Theorem 1.5. Before proceeding, we note that, in most part, the statements follow from simple considerations. In particular, the fact that $\mathbb{C}(X)_{n,l} \subset \ker (T_m)$, for $l \neq -m$, follows from Schur’s orthogonality relations on the circle $S^1$. Furthermore, the fact that $T_m$ acts on the isotypic component $\mathbb{C}(X)_{n,-m}$ as a scalar operator $\lambda_{n,m} \cdot Id$, follows from the fact that $T_m$ is an intertwiner with respect to the $SO(V)$-action, in conjunction with the fact that $\mathbb{C}(X)_{n,-m}$ is an irreducible representation of $SO(V)$ (see Theorem 1.3). The non-trivial statement is the explicit form of the eigenvalues

$$\lambda_{n,m} = \frac{(-1)^{n-m} m}{n (n + 1)} \quad (3.1)$$

for every $n \geq m \geq 1$. The reminder of this section is devoted to the proof of (3.1). We begin by introducing some additional terminology and formulas from representation theory of the group $SO(3)$.

3.1. **Set-up.** We denote by $\mathfrak{so}(3)$ the Lie algebra of the group $SO(3)$ and by $\mathbb{C}\mathfrak{so}(3)$ its complexification. Let $A_1, A_2, A_3 \in \mathfrak{so}(3)$ be infinitesimal generators of
rotations around the $x$, $y$ and $z$ axes respectively:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We introduce the following $sl_2$-triple $(E, F, H)$ in $\mathbb{C}so(3)$:

$$E = iA_2 - A_1,$$

$$F = A_1 + iA_2,$$

$$H = -2iA_3.$$

We denote by $C$ the Casimir element in $\mathcal{U}(so(3))$ - the canonical generator of the center of the universal enveloping algebra. The Casimir can be expressed as

$$C = \frac{1}{4}(H(H+2) + 4EF).$$

We denote by $E_R, F_R, H_R$ the $SO(V)$-invariant vectors fields on $X$, induced from $E, F, H$ respectively via the right action of $SO(3)$. Finally, we denote by $C_R$ the second degree $SO(V)$-invariant differential operator associated with Casimir

$$C_R = \frac{1}{4}(H_R(H_R+2) + 4E_RF_R).$$

### 3.2. Computation of the eigenvalues.

Our computation is based on the following fundamental theorem that establishes a relation between the transport operators of different levels.

**Theorem 3.1.** The following relations hold:

1. For every $m \geq 2$,

   $$E_R \circ T_m \circ F_R = -\frac{m}{m-1} (C_R - m (m - 1)) \circ T_{m-1}.$$

2. For $m = 1$,

   $$E_R \circ T_1 \circ F_R = 1 - \delta_{\tilde{\Delta}_a},$$

where $1$ is the integral operator whose kernel is the constant function 1 and $\delta_{\tilde{\Delta}_a}$ is the integral operator whose kernel is the characteristic function of $\tilde{\Delta}_a = (\pi \times \pi)^{-1}(\Delta_a)$, where $\Delta_a \subset S(V) \times S(V)$ is the anti-diagonal submanifold.

For a proof, see Subsection 3.3.

**Remark 3.2.** Theorem 3.1 extends considerably the development presented in [11] and [12]. In particular, it enables to compute the eigenvalues of all the transport operators, which, for the case of the first level transport operator $T_1$, is a much simpler computation than the one presented in [12].
Granting the validity of Theorem 3.1, we can now prove Formula (3.1). First we note that $E_R$ and $F_R$ act as raising an lowering operators respectively, with respect to the $S^1$-action, that is, they induce isomorphisms of $SO(V)$ representations

$$E_R : \mathbb{C}(X)_{n,l} \cong \mathbb{C}(X)_{n,l+1},$$

$$F_R : \mathbb{C}(X)_{n,l} \cong \mathbb{C}(X)_{n,l-1},$$

for every $n \in \mathbb{N}$ and $l \in \mathbb{Z}$ such that $|l| < n$. The situation is depicted in Figure 6: the isotypic components in $\mathbb{C}(X)$ are represented by the grid points, the fundamental algebra is supported on the dashed diagonal line and the raising and lowering operators $E_R$ and $F_R$ move between the isotypic components in the vertical direction.

The second degree differential operator $E_R F_R$ acts on each isotypic component $\mathbb{C}(X)_{n,l}$ as a scalar operator, since it is an intertwiner with respect to the $SO(V)$-action and $\mathbb{C}(X)_{n,l}$ is irreducible (see Theorem 1.3). Moreover, standard computation reveals that

$$(E_R F_R) | \mathbb{C}(X)_{n,l} = (n - l + 1)(n + l) \cdot Id,$$

which, in particular, implies that

$$(3.4) \quad (E_R F_R) | \mathbb{C}(X)_{n,-m+1} = (n - m + 1)(n + m) \cdot Id,$$

for every $m \geq 0$. Using (3.2), we establish a relation between $\lambda_{n,m}$ and $\lambda_{n,m-1}$, for every $m \geq 2$, as follows: from (3.4), we conclude that

$$(3.5) \quad (E_R \circ T_m \circ F_R) | \mathbb{C}(X)_{n,-m+1} = (n - m + 1)(n + m) \lambda_{n,m} \cdot Id.$$
From (3.2), we conclude that
\[
(E_R \circ T_m \circ F_R) |\mathbb{C}(X)_{n,-m+1} = \frac{-m}{m-1} (C_R - m (m - 1)) \circ T_{m-1} |\mathbb{C}(X)_{n,-m+1}
\]
\[
= \frac{-m}{m-1} (n (n + 1) - m (m - 1)) \lambda_{n,m-1} \cdot Id
\]
\[
= \frac{-m}{m-1} (n - m + 1) (n + m) \lambda_{n,m-1} \cdot Id
\]
where in the second equality we use the fact that the Casimir acts on the \(n\)'th representation of \(SO(V)\) by \(n (n + 1) \cdot Id\). Combining (3.5) with (3.6) we obtain the relation
\[
\lambda_{n,m} = \frac{-m}{m-1} \lambda_{n,m-1},
\]
for every \(m \geq 2\). To finish the computation, we are left to show that
\[
\lambda_{n,1} = \frac{(-1)^{n-1}}{n (n + 1)},
\]
for every \(n \geq 1\). To this end, we use (3.3). Substituting \(m = 1\) in (3.4), we conclude that
\[
(E_R \circ T_1 \circ F_R) |\mathbb{C}(X)_{n,0} = n (n + 1) \lambda_{n,1} \cdot Id.
\]
From (3.3), we conclude that
\[
(E_R \circ T_1 \circ F_R) |\mathbb{C}(X)_{n,0} = (1 - \delta_{X_n}) |\mathbb{C}(X)_{n,0}.
\]
Finally, it is easy to verify that
\[
1 |\mathbb{C}(X)_{n,0} = \begin{cases} 
Id & n = 0 \\
0 & n > 0 
\end{cases},
\]
\[
\delta_{X_n} |\mathbb{C}(X)_{n,0} = (-1)^n \cdot Id, \text{ for every } n \geq 0.
\]
Combining the above with (3.9) and (3.10), we obtain (3.8). This concludes the computation of the eigenvalues of the transport operators.

3.3. Proof of Theorem 3.1. The theorem is proved by direct computation. To this end, we establish concrete formulas for the various differential operators discussed earlier. We begin by identifying \(V\) with \(\mathbb{R}^3\) - choosing some orthonormal basis. Under this identification, \(X = SO(V) = SO(3)\). We introduce the polar coordinates, given by
\[
\text{polar} (\varphi, \theta, \alpha) = e^{\varphi A_3} e^{\theta A_2} e^{\alpha A_3},
\]
for \((\varphi, \theta, \alpha) \in [0, 2\pi) \times [0, \pi] \times [0, 2\pi)\). The vector fields \(E_R, F_R, H_R\) admit the following formulas in the polar coordinates:
\[
E_R = e^{i \alpha} (i \partial_\theta - \cot (\theta) \partial_\alpha + 1/ \sin (\theta) \partial_\varphi),
\]
\[
F_R = e^{-i \alpha} (i \partial_\theta + \cot (\theta) \partial_\alpha - 1/ \sin (\theta) \partial_\varphi),
\]
\[
H_R = 2 i \partial_\alpha.
\]
In addition, we introduce the right invariant vector fields \(E_L, F_L, H_L\) induced from \((E, F, H)\) via the left action of \(SO(3)\) on \(X\). The vector fields \(E_L, F_L, H_L\) admit the following formulas in the polar coordinates:
\[ E_L = -e^{-i\alpha} (i\partial_\theta + \cot (\theta) \partial_\alpha - 1/ \sin (\theta) \partial_\varphi), \]
\[ F_L = -e^{i\alpha} (i\partial_\theta - \cot (\theta) \partial_\alpha + 1/ \sin (\theta) \partial_\varphi), \]
\[ H_L = 2i\partial_\varphi. \]

Next, we observe that the operator \( T_m \) is given by convolution from the right with the distribution kernel \( K_m \in D' (SO(3)) \), defined by
\[ K_m (g) = T_m (g, 1), \]
for every \( g \in SO(3) \). That is
\[ T_m (f) (g) = \int_{h \in SO(3)} f (h) K_m (h^{-1} g) \mu_{Haar}, \]
for every \( f \in C (SO(3)) \). In addition, the operators \( E_R \circ T_m \circ F_R \) and \( C_R \circ T_m \) are given by convolution from the right with the distributions \( -F_L E_R K_m \) and \( C_R K_m \) respectively. In order to prove (3.2) and (3.3), it is enough to show
\[ (3.11) \quad F_L E_R K_m = \frac{-m}{m-1} (C_R - m (m - 1)) K_{m-1}, \]
for every \( m \geq 2 \)
\[ (3.12) \quad F_L E_R K_1 = 1 - \delta_{T_0 J}, \]
respectively, where we recall that \( T_0 \) is our distinguished copy of \( SO(2) \subset SO(3) \), which is also generated by the element \( A_3 \in so(3) \) and \( J \) is our particular matrix in the normalizer \( N(T_0) \), but, in fact, can be chosen to be any matrix in \( N (T_0) - T_0 \).

We have the following explicit formula for the distributions \( K_m, m \geq 1 \), in the polar coordinates:
\[ K_m (\varphi, \theta, \alpha) = \begin{cases} e^{-i m (\varphi + \alpha)} & \theta < \pi, \\ 0 & \theta = \pi. \end{cases} \]

Using the above formula and the formulas for the left/right invariant vector fields associated with the \( sl_2 \)-triple \((E,F,H)\), we obtain the following explicit formulas for \( F_L E_R K_m \) and \( C_R K_m \), for every \( m \geq 2 \).

\[ (3.13) \quad C_R K_m (\varphi, \theta, \alpha) = 2m^2 \left( \frac{1 - \cos (\theta)}{\sin (\theta)^2} \right) K_m, \]
\[ (3.14) \quad F_L E_R K_m (\varphi, \theta, \alpha) = \left( -m^2 + 2m (m - 1) \left( \frac{1 - \cos (\theta)}{\sin (\theta)^2} \right) \right) K_{m-1}. \]

Combining (3.13) with (3.14), we obtain Formula (3.11). Concerning Formula (3.12), we notice, by direct calculation that the distribution \( 1 + F_L E_R K_1 \) is supported on the orbit \( T_0 \cdot J = J \cdot T_0 \), moreover, since it is also \( T_0 \) invariant (both from right and left), it is given by a distribution on the sphere \( S^2 \) supported on the south pole south = \((0,0,-1)\). Hence it is equal to some derivative of \( \delta_{\text{south}} \). In fact, we claim that
\[ 1 + F_L E_R K_1 = \delta_{\text{south}}, \]
which follows from
\[ \int_{SO(3)} (1 + F_L E_{R \mathcal{K}_1}) \mu_{Haar} = 1. \]

This concludes the proof of Theorem 3.1.

4. INVARIANT THEORY OF PLATONIC SUBGROUPS IN SO(V)

In this section we compute the full isotypic decomposition of the algebra \( A \) with respect to each of the three Platonic subgroups \( \Gamma \subset SO(V) \). As a consequence, we derive in each case the dimensions of the various invariant subspaces \( A^m_\Gamma \), for every \( m \geq 0 \). Our development follows closely, unpublished notes of Roger Howe [10], where the strategy is to recast the problem in the set-up of polynomial representations of the group \( SL_2(\mathbb{C}) \).

4.1. Set-up. Let us identify the complex vector space \( W \) equipped with the symmetric form \( r : W \times W \to \mathbb{C} \), with the symmetric power \( S^2(\mathbb{C}^2) \) equipped with the (non-degenerate) symmetric form, induced from the standard symplectic form \( \omega = dx \wedge dy \) on \( \mathbb{C}^2 \). The action of \( SL_2(\mathbb{C}) \) preserves \( r \), thus yields a double cover
\[ SL_2(\mathbb{C}) \xrightarrow{2:1} SO(W). \]

Given a finite subgroup \( \Gamma \subset SO(W) \), we denote by \( \Gamma_1 \) its preimage inside \( SL_2(\mathbb{C}) \) under the the double cover (4.1). The natural inclusion \( W \hookrightarrow S^2(\mathbb{C}^2) \) induces a morphism of algebras \( S^\bullet(W) \to S^\bullet(\mathbb{C}^2) \) that factors through the quotient algebra \( A = S^\bullet(W) / (r^{-1}) \), establishing an isomorphism of \( SL_2(\mathbb{C}) \)-representations
\[ A \xrightarrow{\cong} S^{\text{even}}(\mathbb{C}^2). \]

The above isomorphism maps the homogenous component \( A_m \) isomorphically onto the symmetric power \( S^{2m}(\mathbb{C}^2) \), hence, for any finite subgroup \( \Gamma \subset SO(W) \), we have an isomorphism between the subspaces of invariants
\[ A^\Gamma_m \xrightarrow{\cong} S^{2m}(\mathbb{C}^2)^{\Gamma_1}, \]
for every \( m \geq 0 \).

4.2. Isotypic decompositions. Our goal is to compute the full isotypic decomposition of \( S^m(\mathbb{C}^2) \) with respect to \( \Gamma_1 \), for every \( m \geq 0 \), for the various Platonic subgroups \( \Gamma \subset SO(W) \). We make use of the following general construction: given a finite subgroup \( \Gamma \subset SL_2(\mathbb{C}) \), we associate to it, a graph \( G = G_\Gamma \), whose vertices are the irreducible representations of \( \Gamma \) and there is an edge connecting two irreducible representations \( \pi_1, \pi_2 \in \hat{\Gamma} \) if and only if the dimension \( d_{\pi_1, \pi_2} \) of the vector space of intertwining maps \( \text{Hom}_\Gamma(\pi_1 \otimes \mathbb{C}^2, \pi_2) \) is greater than zero. In that case, we label the edge by \( d_{\pi_1, \pi_2} \). Since the basic representation \( \mathbb{C}^2 \) is self dual (already as a representation of \( SL_2(\mathbb{C}) \)), we have \( d_{\pi_1, \pi_2} = d_{\pi_2, \pi_1} \), hence \( G \) is an undirected graph. In case \( d_{\pi_1, \pi_2} = 1 \), we do not label the edge. The associated graphs of the three Platonic subgroups are summarized in Figure 7; the labels on the vertices specify the dimension of the corresponding irreducible representations.

Given a representation \( \rho \) of \( \Gamma \), it is convenient to describe its isotypic decomposition by labeling each vertex (aka irreducible representation) \( \pi \) of the graph \( G_\Gamma \) with the multiplicity of \( \pi \) in the representation \( \rho \). Figures 8, 9 and 10 summarize the isotypic decompositions of the first few symmetric powers with respect to the tetrahedral, octahedral and icosahedral subgroups respectively.
Based on these calculations, we make the following observations: first we note that each of the Platonic subgroups $\Gamma \subset SO(W)$, their preimage $\Gamma_1 \subset SL_2(\mathbb{C})$ contains the central element $-I \in SL_2(\mathbb{C})$, thus the set of irreducible representations $\widehat{\Gamma}_1$ decomposes into a disjoint union

$$\widehat{\Gamma}_1 = (\widehat{\Gamma}_1)^+ \cup (\widehat{\Gamma}_1)^-,$$

(4.3)

where $(\widehat{\Gamma}_1)^+$ consists of representations that are trivial on $-I$ and $(\widehat{\Gamma}_1)^-$ consists of representations on which $-I$ acts by $-Id$. The elements of $(\widehat{\Gamma}_1)^+$ and $(\widehat{\Gamma}_1)^-$ alternate in the graph $G_{\Gamma_1}$, beginning with the trivial representation in $(\widehat{\Gamma}_1)^+$ located at the first vertex on the left, followed by the standard two dimensional representation $\mathbb{C}^2$ that appears as its right neighbor (this representation is irreducible for all Platonic subgroups). The decomposition (4.3), gives rise to a corresponding decomposition

$$\mathbb{C}(\Gamma_1) = \mathbb{C}(\Gamma_1)^+ \oplus \mathbb{C}(\Gamma_1)^-$$

of the regular representation of $\Gamma_1$ into eigenspaces of the central element $-I$. In addition, we have

$$\mathbb{C}(\Gamma_1)^\pm = \sum_{\pi \in (\Gamma)^\pm} (\dim \pi) \pi.$$

We call $\mathbb{C}(\Gamma_1)^\pm$ the $\pm$-regular representation. We observe that, for any representation $\rho$ of $\Gamma_1$, we have

$$\rho \otimes \mathbb{C}(\Gamma_1) = (\dim \rho) \mathbb{C}(\Gamma_1).$$

Also, if $\rho_1, \rho_2$ are irreducible and $\rho_j \in (\widehat{\Gamma}_1)^{\epsilon_j}$, where $\epsilon_j = +$ or $-$, then $\rho_1 \otimes \rho_2$ will decompose into constituents, all belonging to $(\widehat{\Gamma}_1)^{\epsilon_1 \epsilon_2}$. It follows that if $\rho \in (\widehat{\Gamma}_1)^{\epsilon_1}$, then

$$\rho \otimes \mathbb{C}(\Gamma_1)^{\epsilon_2} = (\dim \rho) \mathbb{C}(\Gamma_1)^{\epsilon_1 \epsilon_2}.$$

In particular, we have

$$\mathbb{C}^2 \otimes \mathbb{C}(\Gamma_1)^\pm = 2 \mathbb{C}(\Gamma_1)^\mp.$$

(4.4)

Finally, we observe that all the representations in $(\widehat{\Gamma}_1)^-$ have even dimension, therefore, we can speak of half the negative regular representation

$$\frac{1}{2} \mathbb{C}(\Gamma_1)^- = \sum_{\pi \in (\Gamma)^-} \left(\frac{\dim \pi}{2}\right) \pi,$$

noting that

$$\dim \frac{1}{2} \mathbb{C}(\Gamma_1)^- = |\Gamma_1|/4.$$
We conclude this subsection with the following propositions:

**Proposition 4.1.** For each of the Platonic subgroups, we have

\[ S^{\Gamma_1/4-1} \mathbb{C}^2 = \frac{1}{2} \mathbb{C}(\Gamma_1)^-. \]

For a proof, see Appendix A

**Proposition 4.2.** For each of the Platonic subgroups, we have

1. \( S^n (\mathbb{C}^2) \oplus S^{\Gamma_1/2-n-2} (\mathbb{C}^2) = \mathbb{C}(\Gamma_1)^{(-1)^n}, \) for every \( 0 \leq n \leq |\Gamma_1|/2 - 2. \)
2. \( S^{\Gamma_1/2-1} (\mathbb{C}^2) = \mathbb{C}(\Gamma_1)^-. \)
3. \( S^{\Gamma_1/2} (\mathbb{C}^2) = \mathbb{C}(\Gamma_1)^+ \oplus S^0 (\mathbb{C}^2). \)
4. \( S^{\Gamma_1/2+n} (\mathbb{C}^2) = \mathbb{C}(\Gamma_1)^{(-1)^n} \oplus S^n (\mathbb{C}^2), \) for every \( n \geq 0. \)

For a proof, see Appendix A.

### 4.3. Dimensions of invariant subspaces

Let us fix a Platonic subgroup \( \Gamma \subset SO(W). \) By Proposition 4.2, statement 4, we conclude that

\[ \dim S^{(||\Gamma_1/2||n+r)} (\mathbb{C}^2)^{\Gamma_1} = \begin{cases} 
1 & r = 0, 6, 8 \\
0 & r = 2, 4, 10 \\
0 & r \text{ odd}
\end{cases} \]

for every \( m \geq 0 \) and \( 0 \leq r < |\Gamma_1|/2, \) where, in the second equality, we use the fact that \( \dim(\mathbb{C}(\Gamma_1)^{-})^{\Gamma_1} = 0 \) and \( \dim(\mathbb{C}(\Gamma_1)^{+})^{\Gamma_1} = 1. \) We proceed to compute the dimensions of the invariant subspaces with respect to \( \Gamma \) in a case by case fashion, noting that, by (4.2), we have

\[ \dim A^\Gamma_m = \dim S^{2m} (\mathbb{C}^2)^{\Gamma_1}. \]

#### 4.3.1. Tetrahedral subgroup

In the case \( \Gamma \) is a tetrahedral subgroup, we have \( |\Gamma_1|/2 = 12. \) Inspecting Figure 8, we deduce

\[ \dim S^r (\mathbb{C}^2)^{\Gamma_1} = \begin{cases} 
1 & r = 0, 6, 8 \\
0 & r = 2, 4, 10 \\
0 & r \text{ odd}
\end{cases} \]

for any \( 0 \leq r < 12. \) Hence

\( \dim A^\Gamma_{6n+r} = \begin{cases} 
1 & r = 0, 3, 4 \\
0 & r = 1, 2, 5
\end{cases} \)

for any \( m \geq 0 \) and \( 0 \leq r < 6. \)

#### 4.3.2. Octahedral subgroup

In the case \( \Gamma \) is an octahedral subgroup, we have \( |\Gamma_1|/2 = 24. \) Inspecting Figure 9, we deduce

\[ \dim S^r (\mathbb{C}^2)^{\Gamma_1} = \begin{cases} 
1 & r = 0, 8, 12, 16, 18, 20 \\
0 & r = 2, 4, 6, 10, 14, 22 \\
0 & r \text{ odd}
\end{cases} \]

for any \( 0 \leq r < 24. \) Hence

\( \dim A^\Gamma_{12n+r} = \begin{cases} 
1 + 1 & r = 0, 4, 6, 8, 9, 10 \\
0 & r = 1, 2, 3, 5, 7, 11
\end{cases} \)

for any \( n \geq 0 \) and \( 0 \leq r < 12. \)
4.3.3. *Icosahedral subgroup.* In the case \( \Gamma \) is icosahedral subgroup, we have \( |\Gamma_1|/2 = 60 \). Inspecting Figure 10, we deduce, using Proposition 4.2, statements 1 and 2, that

\[
\dim S^r (\mathbb{C}^2)^{\Gamma_1} = \begin{cases} 
1 & r = 0, 12, 20, 24, 30, 32, 36, 40, 42, 44, 48, 50, 52, 54, 56 \\
0 & r = 2, 4, 6, 8, 10, 14, 16, 18, 22, 26, 28, 34, 38, 46, 58 \\
0 & r \text{ odd}
\end{cases},
\]

for any \( 0 \leq r < 60 \). Hence

\[
(4.7) \quad \dim A^r_{30n+r} = \begin{cases} 
n + 1 & r = 0, 6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28 \\
n & r = 1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 14, 17, 19, 23, 29
\end{cases},
\]

for any \( n \geq 0 \) and \( 0 \leq r < 30 \).

**Appendix A. Proofs**

A.1. **Proof of Proposition 1.2.** Notice that, an automorphism of \( A \), preserving the grading is induced by a linear isomorphism \( T : W \to W \), preserving the metric \( r \) up to a scalar. Hence \( T \) is an element of the group of orthogonal similitudes \( \text{GSO}(W) = \text{SO}(W) \times \mathbb{C}^\times \). This proves that \( \text{Aut}(A) = \text{SO}(W) \times \mathbb{C}^\times \).

Next, an element \((g, \lambda) \in \text{SO}(W) \times \mathbb{C}^\times\) that respects the real structure must preserve the real vector space \( V = W^{\text{conj}} \), hence \( g \in \text{SO}(V) \) and \( \lambda \in \mathbb{R}^\times \). This proves that \( \text{Aut}_{\mathbb{R}}(A) = \text{SO}(V) \times \mathbb{R}^\times \).

Finally, an element \((g, \lambda) \in \text{SO}(V) \times \mathbb{R}^\times\) that preserves the Hermitian form \( h \) must satisfy \( \lambda \in \mu_2 \). This proves that \( \text{Aut}_{\mathbb{R},h}(A) = \text{SO}(V) \times \mu_2 \).

This concludes the proof of the proposition.

A.2. **Proof of Theorem 1.3.** The basic observation is, that \( \mathbb{C}(X) \), as a representation of \( \text{SO}(V) \times \text{SO}(3) \), admits the following isotypic decomposition

\[
\mathbb{C}(X) = \bigoplus_{n=0}^{\infty} V_n \otimes U_n,
\]

where \( V_n \) is the unique irreducible representation of \( \text{SO}(V) \) of dimension \( 2n + 1 \), and, similarly, \( U_n \) is the unique irreducible representation of \( \text{SO}(3) \) of dimension \( 2n + 1 \). This assertion, principally, follows from the Peter Weyl Theorem for the regular representation of \( \text{SO}(3) \). This implies that the isotypic decomposition of \( \mathbb{C}(X)_m \) takes the following form

\[
\mathbb{C}(X)_m = \bigoplus_{n=0}^{\infty} V_n \otimes U_n^m,
\]

where \( U_n^m \) is the weight \( m \) space with respect to the action of \( T_0 \subset \text{SO}(3) \). The statement now follows from the following standard fact about the weight decomposition:

\[
\dim U_n^m = \begin{cases} 
0 & n < m \\
1 & n \geq m
\end{cases}.
\]

This concludes the proof of the Theorem.
A.3. **Proof of Theorem 1.4.** We know that the morphism $\delta : X \to Y$ is a morphism of $SO(V) \times S^1$ spaces. From the fact that $\delta$ commutes with the $S^1$-action we get that $\delta^* (A_m) \subset C(X)_{-m}$. From the fact that $\delta$ commutes by the $SO(V)$-action, we get that $\delta^* (A_m) = C(X)_{m-m}$, since, by the theory of spherical harmonics, $A_m$ is the irreducible representation of $SO(V)$ of dimension $2m+1$ and $C(X)_{m-m}$ is the only copy of this representation in $C(X)_{-m}$.

This concludes the proof of the theorem.

A.4. **Proof of Proposition 1.8.** We assume that $Fr^*$ induces isomorphisms of real vector spaces

\begin{align}
Fr^* : A_1 &\cong Fr^* A_1, \\
Fr^* : A_2 &\cong Fr^* A_2.
\end{align}

These conditions hold, for example, if the number of projection images is large enough, assuming $\text{Im} Fr \subset X$ are distributed uniformly at random. Under these conditions, $\text{Sym}_2 (Fr^*)$ restricts to give an isomorphism of one dimensional real vector spaces

\begin{align}
\text{Sym}_2 (Fr^*) : Q &\cong Q,
\end{align}

where $Q = \ker (m_{A_1})$ and $m_{A_1} : \text{Sym}_2 A_1 \to A_2$ is the map induced from the multiplication mapping on $C(X)$.

The isomorphism $\tau$ is defined as follows: first, consider the morphism of graded algebras

$$\text{Sym}_\bullet (\tau_1) : \text{Sym}_\bullet (W^*) = \text{Poly} (W) \to \hat{A},$$

induced from the linear map $\tau_1 = Fr^* \circ \delta^* : W^* \to Fr^* A_1$. The morphism $\text{Sym}_\bullet (\tau_1)$ is a morphism of real graded algebras, since $\tau_1$ is a morphism of real vector spaces. Furthermore, $\text{Sym}_\bullet (\tau_1)$ is surjective, since $\tau_1$ is a linear isomorphism. The latter follows from (A.1) in conjunction with the fact that $\delta^* : W^* \to A_1$ is a linear isomorphism. Since $\delta^* (r) \in A_1$, we conclude from (A.3) that $\text{Sym}_\bullet (\tau_1) \in Q$ and that $\text{Sym}_\bullet (\tau_1)$ yields an isomorphism of real graded algebras

$$\tau : A = \text{Sym}_\bullet (W^*) / \langle r \rangle \to \hat{A}.$$ 

Finally, the fact that $\tilde{\Phi} (I) \circ \tau = \Phi (I)$, for every $I \in P$, follows directly from the construction.

This concludes the proof of the proposition.

A.5. **Proof of Theorem 2.2.** We proceed in case by case fashion. We use the fact that the algebra of invariants has no zero divisors. One way to see this, is by realizing the algebra of invariants as a subalgebra of $S^\text{even} (C^2)$ (see Section 4).

A.5.1. **Cyclic subgroups.** Let $\Gamma \simeq C_l$, $l \geq 2$. We are allowed to assume that $\Gamma$ is the group consisting of elements of order $l$, $l \geq 2$, in some maximal (algebraic) torus $T \subset SO(V)$. The idea is to use the weight decomposition of $A$ with respect to the torus $T$. In more details, the $m$'th homogenous component $A_m$, admits a weight decomposition

\begin{align}
A_m = \bigoplus_{k=-m}^{m} A^k_m,
\end{align}

where $A^k_m$ is the irreducible representation of $SO(V)$ of dimension $2k+1$ and $C(X)_{m-k}$ is the only copy of this representation in $C(X)_{-k}$. This concludes the proof of the proposition.
where \( \dim_k^k = 1 \). In terms of this decomposition, the \( \Gamma \)-invariant subspace \( A_m^\Gamma \), is given by

\[
A_m^\Gamma = \bigoplus_{k=-m}^{m} A_k^m.
\]

Consider invariant polynomials \( Q_1 \in A_1^0 \) and \( Q_l \in A_l^1, Q_{-l} \in A_l^{-1} \). The algebra \( A^\Gamma \) is generated by these three polynomials, since given integers \( m \geq 1 \) and \(-m \leq k \leq m\), such that \( l \mid k \), the weight space \( A_k^m \) is spanned by the polynomial \( Q_m^{m-k} Q_k^{k/l} \), in case \( k \geq 0 \) and by the polynomial \( P_m^{m-k} Q_{-l}^{k/l} \), in case \( k < 0 \). The polynomials \( Q_m^{m-k} Q_k^{k/l} \) and \( Q_m^{m-k} Q_{-l}^{k/l} \) are non-zero since \( A^\Gamma \) has no zero divisors. To obtain real generators, we take

\[
P_1 = Q_1, \\
P_l = Q_l + Q_{-l}, \\
P_l' = Q_l - Q_{-l}.
\]

To verify the relation, we identify the Euclidean vector space \( V \) with the standard Euclidean vector space \( \mathbb{R}^3 \), let \( T \) consists of rotations around the \( z \)-axis and take

\[
P_1 = z, \\
P_l = (x - iy)^l + (x + iy)^l, \\
P_l' = (x + iy)^l - (x + iy)^l,
\]

where \( x, y, z \in V^* \) are the standard coordinate functions. Granting these choices, we have

\[
P_l^2 + P_l'^2 - 2(-1)^l P_l^2 = 2 \left( (x^2 + y^2)^l - (-z)^l \right) = 0 \mod x^2 + y^2 + z^2.
\]

The above expression must be the unique relation, since the Krull dimension of \( A^\Gamma \) is 2. This concludes the description of the algebra of invariants for the case of cyclic subgroups.

A.5.2. Dihedral subgroups. Let \( \Gamma \cong D_l, l \geq 2 \). The group \( \Gamma \) contains a maximum normal subgroup \( \Gamma_0 \vartriangleleft \Gamma \), isomorphic to the cyclic group \( C_l \). We are allowed to assume that \( \Gamma_0 \) consists of all elements of order \( l \) in some maximal torus \( T \subset SO(V) \) and \( \Gamma \) is the group generated by \( \Gamma_0 \) and an element \( \theta \in N(T) - T \). In order to describe the algebra of invariants, we use again the weight decomposition (A.4) with respect to \( T \).

The element \( \theta \) induces isomorphisms

\[
\theta : A_m^k \cong A_{-m}^{-k},
\]

for every integer \( m \geq 0 \) and \(-m \leq k \leq m\). Furthermore, it is easy to verify that \( \theta|_{A_m^0} = (-1)^m \Id \). Based on these two facts, we conclude

\[
\dim A_m^\Gamma = \begin{cases} 
  n & m \text{ odd} \\
  n + 1 & m \text{ even}
\end{cases}.
\]
for every integer \( m \geq 0 \), taking the form \( m = nl + r \), \( 0 \leq r < l \). Taking invariant polynomials

\[
\begin{align*}
P_2 & \in A_2^0, \\
P_l & \in (A_l^1 \oplus A_l^{-1})^\theta, \\
P_{l+1} & \in (A_{l+1}^1 \oplus A_{l+1}^{-1})^\theta,
\end{align*}
\]

we claim that \( p_2, p_l \) and \( p_{l+1} \) generate \( A^\Gamma \). To show this, it’s enough to verify that

\[
A^\Gamma_{(n+1)l+r} = \underbrace{P_2 \cdot A^\Gamma_{(n+1)l+r-2}}_{V_1} + \underbrace{P_l \cdot A^\Gamma_{nl+r} + P_{l+1} \cdot A^\Gamma_{nl+r-1}}_{V_2},
\]

which, in turns, follows from dimension considerations, using (A.6) in conjunction with the fact that the vector spaces \( V_1, V_2 \subset A^\Gamma_{(n+1)l+r} \) are not equal.

In order to verify the relation, we identify the Euclidean vector space \( V \) with the standard Euclidean vector space \( \mathbb{R}^3 \), let \( T \) consists of rotations around the \( z \)-axis, take

\[
\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

and, finally, realize the generators as

\[
\begin{align*}
P_2 &= z^2, \\
P_l &= (x + iy)^l + (x - iy)^l, \\
P_{l+1} &= z \left( (x + iy)^l - (x - iy)^l \right).
\end{align*}
\]

Granting these choices, we have

\[
P_2 \cdot P_{l+1}^2 - P_l^2 - 2 (-1)^l \cdot P_{l+1}^2 = 2z^2 \left( (x^2 + y^2)^l - (-z^2)^l \right) = 0 \mod x^2 + y^2 + z^2.
\]

The above expression must be the unique relation, since the Krull dimension of \( A^\Gamma \) is 2. This concludes the description of the algebra of invariants for the case of dihedral subgroups.

A.5.3. **Tetrahedral subgroup.** Inspecting Figure 8, we see that the isotypic decomposition of \( A_2 \) consists of two non-trivial one dimensional representations; let \( \psi_2, \overline{\psi}_2 \in A_2 \) be corresponding eigenvectors. We have

\[
\overline{\psi}_2 = \text{conj} (\psi_2).
\]

Evidently, the polynomials \( \psi_2, \overline{\psi}_2 \) are algebraically independent, thus generate a polynomial algebra in 2 generators. In this algebra, the invariants forms a subalgebra generated by \( \psi_2 \overline{\psi}_2, \psi_2 \psi_3, \overline{\psi}_2 \psi_3 \) of degrees 4 and 6 respectively. By Formula (4.5), we see that these account for all the invariants of degree up-to 6, except of an invariant \( \varphi_3 \) of degree 3, which can be taken to be real. In particular, we can express \( \varphi_3^2 \) as a linear combination of \( \psi_2^3 \) and \( \overline{\psi}_2^3 \):

\[
\varphi_3^2 = c \psi_2^3 + \overline{c} \overline{\psi}_2^3.
\]
By Formula (4.5), we conclude that \( A^\Gamma_{6n} \) is spanned by \( \psi_2^{3k} \overline{\psi}_2^{3(n-k)} \), for \( 0 \leq k \leq n \) and that

\[
\begin{align*}
A^\Gamma_{6n+3} &= \varphi_3 A^\Gamma_{6n}, \quad A^\Gamma_{6n+4} = \psi_2 \overline{\psi}_2 A^\Gamma_{6n}, \quad A^\Gamma_{6n+7} = \psi_2 \overline{\psi}_2 \varphi_3 A^\Gamma_{6n}, \\
A^\Gamma_{6n+8} &= (\psi_2 \overline{\psi}_2)^2 A^\Gamma_{6n}, \quad A^\Gamma_{6n+11} = (\psi_2 \overline{\psi}_2)^2 \varphi_3 A^\Gamma_{6n},
\end{align*}
\]

for every \( n \geq 0 \). This implies that \( A^\Gamma \) is generated by the polynomials \( \psi_2 \overline{\psi}_2, \psi_2^3, \overline{\psi}_2^3 \) and \( \varphi_3 \). Now, define \( P_3 = \varphi_3, P_4 = \psi_2 \overline{\psi}_2 \) and \( P_6 = i (\psi_2^3 - \overline{\psi}_2^3) \). Direct verification shows that \( A^\Gamma \) is generated by the polynomials \( P_3, P_4 \) and \( P_6 \) and these polynomials satisfy a relation in degree 12 of the form

\[
aP_3^2 + bP_4^3 + cP_6^2 = 0.
\]

This concludes the description of the algebra of invariants for the case of the tetrahedral subgroup.

A.5.4. Octahedral subgroup. The octahedral group contains the tetrahedral group as a normal subgroup of index 2, thus its invariants are a subalgebra of the invariants for the tetrahedral group. By comparing the isotypic decompositions for \( A_2 \) and \( A_6 \) for the octahedral group (Figure 9) and tetrahedral cases (Figure 8), we see that the octahedral group interchanges between \( \psi_2 \) and \( \overline{\psi}_2 \) and that \( \varphi_3 \) is character vector with respect to the unique non-trivial quadratic character of the octahedral group. Therefore, \( P_4 = \psi_2 \overline{\psi}_2 \) is the invariant of degree 4, \( P_6 = \varphi_3^2 \) is the invariant of degree 6 and \( P_9 = \varphi_3 (\psi_2^3 - \overline{\psi}_2^3) \) is the invariant of degree 9. The three invariants are real polynomials and, evidently, the polynomials \( P_4, P_6 \) are algebraically independent. Direct verification, using Formula (4.6), reveals that \( A^\Gamma \) is generated by \( P_4, P_6, P_9 \). Furthermore these polynomials satisfy a relation in degree 18 of the form

\[
aP_4^2 + bP_6^3 + cP_9^2 = 0,
\]

since \( \dim A^\Gamma_{18} = 2 \). This concludes the description of the algebra of invariants for the case of the octahedral subgroup.

A.5.5. Icosahedral subgroup. Inspecting Formula 4.7, we see that the icosahedral invariants in degrees up to 15 occur in degrees 6, 10, 12 and 15. Denote these in degrees 6, 10 and 15 by \( P_6, P_{10} \) and \( P_{15} \) respectively. Evidently, \( P_6^2 \) is the invariant of degree 12. Monomials in these will produce invariants in degrees 16, 18, 20, 21, 22, 24, 25, 26, 27, 28. From Proposition 4.2, statement 1, we see that these account for all invariants of degree less that 30. In degree 30 there are two invariants which implies a relation between \( P_6^2, P_{10}^3 \) and \( P_{15}^2 \) of the form

\[
aP_6^2 + bP_{10}^3 + cP_{15}^2 = 0.
\]

In addition, direct verification, using Formula (4.7), reveals that \( A^\Gamma \) is generated by \( P_6, P_{10}, P_{15} \). This concludes the description of the algebra of invariants for the case of the icosahedral subgroup.

A.6. Proof of Theorem 2.3. We focus our attention on proving that \( \text{Aut}_\mathbb{R}(\overline{\Omega}) = W(\Gamma) \times \mathbb{R}^2 \), for the various finite subgroups \( \Gamma \). The proof of the other case follows along similar lines, thus it is omitted. We note that, the group \( \text{Aut}_\mathbb{R}(\overline{\Omega}) \) coincides with the group \( \text{Aut}_\mathbb{R}(\overline{\mathbb{A}}) \), consisting of automorphisms of the algebra \( \overline{\mathbb{A}} = A^\Gamma \), preserving the grading and the real structure. We proceed with case by case study.
A.6.1. Tetrahedral subgroup. Consider the three generators \( P_3, P_4 \) and \( P_6 \) of \( \mathcal{G} \), specified by the structure theorem (Theorem 2.2). The normalizer \( N(\Gamma) \) is the octahedral subgroup, hence the quotient group \( W(\Gamma) \) consists of two elements: the unit element and an order two element \( \theta \), which acts by

\[
\theta P_3 = -P_3, \\
\theta P_4 = P_4, \\
\theta P_6 = -P_6,
\]

where the first condition follows from the fact that \( \dim A_3^{N(\Gamma)} = 0 \), the second condition follows from the fact that \( \dim A_4^{N(\Gamma)} = \dim A_4^\Gamma = 1 \) and the third condition follows from the fact that \( \dim A_6^{N(\Gamma)} = \dim A_6^\Gamma + 1 \).

We proceed to show that every automorphism \( \varphi \in \Aut_R(\mathcal{G}) \) coincides with the action of some element in \( W(\Gamma) \times \mathbb{R}^\times \). Given an automorphism \( \varphi \in \Aut_R(\mathcal{G}) \), it must satisfy

\[
\varphi(P_3) = \lambda_3 P_3, \\
\varphi(P_4) = \lambda_4 P_4, \\
\varphi(P_6) = \lambda_6 P_6 + \mu_6 P_3^2,
\]

for some \( \lambda_3, \lambda_4, \lambda_6 \in \mathbb{R}^\times \), where the first and second conditions follow from the fact that the corresponding homogenous components are one dimensional and the third condition follows from the fact that the degree 6th component is two dimensional, spanned by \( P_6 \) and \( P_3^2 \) (see Proposition 2.4).

In addition, the relation \( aP_3^2 + bP_4^3 + cP_6^2 = 0 \) in conjunction with the fact that the polynomials \( P_3^2, P_4^3, P_6^2 \) and \( P_3^2 P_6 \) span the three-dimensional vector space \( \mathcal{G}_{12} \) (see Proposition 2.4) imply that

(A.7)

\[
\mu_6 = 0, \\
\lambda_6^2 = \lambda_3^4, \\
\lambda_4^3 = \lambda_3^3.
\]

Letting \( \lambda_3 = t^3 \), for some \( t \in \mathbb{R}^\times \), we get from (A.7) that \( \lambda_6 = \pm t^6 \) and \( \lambda_4 = t^4 \). If \( \lambda_6 = t^6 \), then \( \varphi \) coincides with the action of the element \( (1, t) \in W(\Gamma) \times \mathbb{R}^\times \). If \( \lambda_6 = -t^6 \), then \( \varphi \) coincides with the action of the element \( (\theta, -t) \in W(\Gamma) \times \mathbb{R}^\times \).

This concludes the proof of the rigidity property for the Tetrahedral subgroup.

A.6.2. Octahedral subgroup. The normalizer \( N(\Gamma) = \Gamma \), hence the quotient group \( W(\Gamma) \) is trivial. We proceed to show that every automorphism \( \varphi \in \Aut_R(\mathcal{G}) \) coincides with the action of some element in \( W(\Gamma) \times \mathbb{R}^\times \). An automorphism \( \varphi \in \Aut_R(\mathcal{G}) \) must satisfy

\[
\varphi(P_3) = \lambda_3 P_3, \\
\varphi(P_4) = \lambda_6 P_6, \\
\varphi(P_9) = \lambda_9 P_9,
\]

for some \( \lambda_3, \lambda_6, \lambda_9 \in \mathbb{R}^\times \), where \( P_3, P_6, P_9 \) are the three generators specified in the structure theorem (Theorem 2.2). The conditions follow from the fact that the corresponding homogenous components are one dimensional.
In addition, the relation \( aP_2^2 + bP_3^3 + cP_4^3P_6 = 0 \), implies that \( \lambda_6^3 = \lambda_6^2 = \lambda_6^3 \lambda_6 \), which, in turns, implies that
\[
\begin{align*}
\lambda_9^2 &= \lambda_6^3, \\
\lambda_4^3 &= \lambda_6^2.
\end{align*}
\]

Letting \( \lambda_6 = t^6 \), for some \( t \in \mathbb{R}^\times \), we get from (A.8) that \( \lambda_6 = t^6 \) and \( \lambda_4 = t^4 \). This implies that \( \varphi \) coincides with the action of the element \( t \in \mathbb{R}^\times = W(\Gamma) \times \mathbb{R}^\times \).

This concludes the proof of the rigidity property for the Octahedral subgroup.

A.6.3. Icosahedral subgroup. The normalizer \( N(\Gamma) = \Gamma \), hence the quotient group \( W(\Gamma) \) is trivial. We proceed to show that every automorphism \( \varphi \in \text{Aut}_R(\overline{A}) \) coincides with the action of some element in \( W(\Gamma) \times \mathbb{R}^\times \). An automorphism \( \varphi \in \text{Aut}_R(\overline{A}) \) must satisfy
\[
\begin{align*}
\varphi(P_6) &= \lambda_6 P_6, \\
\varphi(P_{10}) &= \lambda_{10} P_{10}, \\
\varphi(P_{15}) &= \lambda_{15} P_{15},
\end{align*}
\]
for some \( \lambda_6, \lambda_{10}, \lambda_{15} \in \mathbb{R}^\times \), where \( P_6, P_{10}, P_{15} \) are the three generators specified in the structure theorem (Theorem 2.2). The conditions follow from the fact that the corresponding homogenous components are one dimensional.

In addition, the relation \( aP_{15}^2 + bP_{10}^3 + cP_6^6 = 0 \), implies that \( \lambda_{10}^3 = \lambda_{15}^2 = \lambda_6^5 \). Letting \( \lambda_{15} = t^{15} \), for some \( t \in \mathbb{R}^\times \), we get from (A.9) that \( \lambda_6 = t^6 \) and \( \lambda_{10} = t^{10} \). This implies that \( \varphi \) coincides with the action of the element \( t \in \mathbb{R}^\times = W(\Gamma) \times \mathbb{R}^\times \).

This concludes the proof of the rigidity property for the Icosahedral subgroup.

A.6.4. Cyclic subgroups. Let \( \Gamma \simeq C_l \), \( l \geq 2 \). The group \( \Gamma \) coincides with the subgroup of elements of order \( l \) in some maximal torus \( T \subset SO(V) \), noting that \( T \simeq SO(2) \), hence the normalizer \( N(\Gamma) = N(T) \simeq O(2) \). Consider the real polynomials \( P_1, P_l \) and \( P'_l \), generating the algebra \( \overline{A} \) according to the structure theorem (Theorem 2.2). A simple inspection reveals that the action of \( N(\Gamma) \) on the homogenous component \( \overline{A}_l \), induces a surjection
\[
N(\Gamma) \rightarrow O(\mathbb{R}P_l \oplus \mathbb{R}P'_l).
\]

We proceed to show that every automorphism \( \varphi \in \text{Aut}_R(\overline{A}) \) coincides with the action of some element in \( W(\Gamma) \times \mathbb{R}^\times \). An automorphism \( \varphi \in \text{Aut}_R(\overline{A}) \) must satisfy
\[
\begin{align*}
\varphi(P_1) &= \lambda P_1, \\
\varphi(P_l) &= aP_1 + bP'_l + \mu_1 P'_1, \\
\varphi(P'_l) &= cP_1 + dP'_l + \mu_2 P'_1,
\end{align*}
\]
where \( \lambda \in \mathbb{R}^\times \) and \( a, b, c, d, \mu_1, \mu_2 \in \mathbb{R} \). The first condition follows from the fact that \( \overline{A}_1 \) is one dimensional, spanned by \( P_1 \) and the second and third conditions follow from the fact that \( \overline{A}_l \) is three-dimensional, spanned by \( P_1, P'_l \) and \( P'_l \). Expanding
the relation $\varphi(P_2)^2 + \varphi(P_2) - (-1)^l \varphi(P_2)^2 = 0$, we obtain

\begin{equation}
(A.11) \quad 0 = (a^2 + c^2) P_2^2 + (b^2 + d^2) P_4^2 - (-1)^l \lambda^l P_2^{2l}
\end{equation}

\[ + (\mu_1 c + \mu_2 a) P_4^2 P_2 + (\mu_1 d + \mu_2 b) P_4^2 P_2 + (ab + cd) P_4^2 P_2, \]

where $V_1, V_2$ and $V_3$ denote the subspaces of $\mathcal{A}_{2l}$ spanned by the polynomials \{\(P_2^2, P_4^2, P_6^2\), \{\(P_4^2 P_2, P_4^2 P_2\)\} and the polynomial $P_4^2 P_2$ respectively. A simple inspection reveals that $\dim V_1 = 2$, $\dim V_2 = 2$ and $\dim V_3 = 1$ and, moreover, the intersection between each pair of subspaces is zero. Consequently, Equation (A.11) in conjunction with the relation $P_2^2 + P_4^2 - (-1)^l P_2^{2l} = 0$ imply the following conditions:

\begin{align*}
a^2 + c^2 &= \lambda^{2l}, \\
b^2 + d^2 &= \lambda^{2l}, \\
ab + cd &= 0, \\
\mu_1 &= 0, \\
\mu_2 &= 0.
\end{align*}

The above conditions imply that $\varphi$ preserves the subspace $\mathbb{R}P_2 \oplus \mathbb{R}P_4$ and that $\lambda^{-1} \varphi \in O(\mathbb{R}P_2 \oplus \mathbb{R}P_4)$. Pick an element $g \in N(\Gamma)$ which is a preimage of $\lambda^{-1} \varphi$ under the surjection (A.10). The automorphism $\varphi$ coincides with the action of the element $(g, \lambda) \in W(\Gamma) \times \mathbb{R}^\times$.

This concludes the proof of the rigidity property for cyclic subgroups.

A.6.5. 

Dihedral subgroups $l \geq 3$. Let $\Gamma \simeq D_l, l \geq 3$. Since $l \geq 3$, the normalizer subgroup $N(\Gamma)$ is isomorphic to the dihedral group of order $2l$. Consequently, the quotient group $W(\Gamma)$ consists of two elements: the unit element and an element $\theta$ of order two. Let $P_2, P_4$ and $P_{l+1}$ be the generators of the algebra $\mathcal{A}$, according to the structure theorem (Theorem 2.2). A simple inspection reveals that

\begin{align*}
\theta P_2 &= P_2, \\
\theta P_4 &= -P_4, \\
\theta P_{l+1} &= -P_{l+1}.
\end{align*}

We proceed to show that every automorphism $\varphi \in \text{Aut}_\mathbb{R}(\mathcal{A})$ coincides with the action of some element in $W(\Gamma) \times \mathbb{R}^\times$, for the case $l$ is even, noting that the case of $l$ odd is treated in similar manner. An automorphism $\varphi \in \text{Aut}_\mathbb{R}(\mathcal{A})$ must satisfy

\begin{align*}
\varphi(P_2) &= \lambda_2 P_2, \\
\varphi(P_4) &= \lambda_4 P_4 + \mu_4 P_4^{l/2}, \\
\varphi(P_{l+1}) &= \lambda_{l+1} P_{l+1},
\end{align*}

where $\lambda_2, \lambda_{l+1} \in \mathbb{R}^\times$, and $\lambda_4, \mu_4 \in \mathbb{R}$. The first and third conditions follow from the fact that the corresponding homogenous components are one dimensional and the second condition follows from the fact that $\mathcal{A}$ is two dimensional, spanned by the polynomials $P_4$ and $P_4^{l/2}$. Expanding the relation $\varphi(P_2) \varphi(P_4)^2 + \varphi(P_{l+1})^2 -
\[ 2(-1)^l \varphi (P_2)^{l+1} = 0, \] we obtain

\[ (A.12) \quad 0 = \lambda_2 \lambda_2^2 P_2 P_2^2 + \lambda_{l+1}^2 P_{l+1}^2 + \left( \mu_l^2 - 2(-1)^l \lambda_2^{l+1} \right) P_{l+1}^2 + \mu_{l+1} \lambda_2 P_l P_{2/l+1}^2. \]

where \( V_1 \) is the two dimensional subspace of \( \mathbb{A}_{2l+2} \), spanned by \( \{ P_2^2, P_{l+1}^2, P_{2/l+1}^2 \} \) and \( V_2 \) is the subspace of \( \mathbb{A}_{2l+2} \) spanned by \( P_{l+1}^2 \). A simple inspection reveals that \( V_1 \cap V_2 = 0 \). Consequently, Equation (A.12) in conjunction with the relation \( P_2 P_2^2 + P_{l+1}^2 - 2(-1)^l P_{2/l+1}^2 = 0 \) imply the following conditions:

\[ (A.13) \begin{align*}
\lambda_l^2 &= \lambda_2^2, \\
\lambda_{l+1}^2 &= \lambda_2^{l+1}, \\
\mu_l &= 0.
\end{align*} \]

Letting \( \lambda_{l+1} = t^{l+1} \), for some \( t \in \mathbb{R}^\times \), noting that \( l+1 \) is odd. Conditions (A.13) imply that \( \lambda_l = \pm t^l \) and \( \lambda_2 = t^2 \). If \( \lambda_l = t^l \) then \( \varphi \) coincides with the action of the element \( (1, t) \in W(\Gamma) \times \mathbb{R}^\times \) and if \( \lambda_l = -t^l \) then \( \varphi \) coincides with the action of the element \( (\theta, -t) \in W(\Gamma) \times \mathbb{R}^\times \).

This concludes the proof of the rigidity property in case \( \Gamma \) is a dihedral subgroup of order \( \geq 3 \).

A.7. **Proof of Theorem 2.5.** The fact that the "isotypic" component \( C(X)_{n,l} \subset \ker T_m \), for every \( l \neq -m \), follows from Schur’s orthogonality relations on the circle. We are left to show that the invariant transport operator \( T_m \) acts as \( \lambda_{n,m} \cdot Id \) on the "isotypic" component \( C(X)_{n,-m} \), where \( \lambda_{n,m} \) is the eigenvalue of the plain transport operator \( T_m \) when acting on the isotypic component \( C(X)_{n,-m} \).

Let \( \pi: X \to \overline{X} \) denote the canonical projection. The statement follows from the general relation

\[ (A.14) \quad \pi^* \circ T_m = T_m \circ \pi^*, \]

which we proceed to verify. Let \( f \in C(\overline{X}) \) and fix a point \( x \in X \). We have to show

\[ (A.15) \quad T_m (f) (\pi (x)) = T_m (f) (x). \]
Let \( X_0 \subset X \) be a fundamental domain with respect to the \( \Gamma \)-action. Developing the left hand side of (A.15), we obtain

\[
\mathcal{T}_m (f)(\pi(x)) = \int_{y \in X} \mathcal{T}_m (\pi(x), \pi(y)) \pi^* f(y) \mu_{\text{Haar}})
\]

(A.16)

\[
= \int_{y \in X} \mathcal{T}_m (\pi(x), \pi(y)) \pi^* f(y) \mu_{\text{Haar}}
\]

\[
= |\Gamma| \cdot \int_{y \in X_0} \mathcal{T}_m (\pi(x), \pi(y)) \pi^* f(y) \mu_{\text{Haar}}
\]

\[
= |\Gamma| \cdot \int_{y \in X_0} \frac{1}{|\Gamma|} \sum_{y' \in \pi(y)} \mathcal{T}_m (x, y') \pi^* f(y') \mu_{\text{Haar}}
\]

\[
= \int_{y' \in X} \mathcal{T}_m (x, y') \pi^* f(y') \mu_{\text{Haar}} = \mathcal{T}_m (f)(x),
\]

where the first equality is the definition of \( \mathcal{T}_m \), the second equality follows from the definition of the push-forward measure and the forth equality follows from the definition of the invariant transport kernel.

This concludes the proof of the theorem.

A.8. Proof of Proposition 4.1. The statement is verified by inspection of the calculated structure of \( S_5 \) in the case of the tetrahedral group (see Figure 8), \( S_{11} \) in the case of the octahedral group (see Figure 9) and \( S_{29} \) in the case of the icosahedral group (see Figure 10).

This concludes the proof of the proposition.

A.9. Proof of Proposition 4.2. The proof of the first statement proceeds by induction on \( N = |n - (|\Gamma_1|/4 - 1)| \). For \( n = |\Gamma_1|/4 - 1 \), the statement follows from Proposition 4.1, establishing the basis for the induction. Assuming the statement holds for every \( n \) such that \( |n - (|\Gamma_1|/4 - 1)| \leq N \), let \( n \) satisfy \( |n - (|\Gamma_1|/4 - 1)| = N + 1 \). We write

\[
2\mathbb{C} (\Gamma_1)^{(-1)^n+1} = \mathbb{C}^2 \otimes \mathbb{C} (\Gamma_1)^{(-1)^n}
\]

\[
= \mathbb{C}^2 \otimes S^n (\mathbb{C}^2) \oplus \mathbb{C}^2 \otimes S^{\lceil |\Gamma_1|/2 \rceil - n - 2} (\mathbb{C}^2)
\]

\[
= (S^{n+1} (\mathbb{C}^2) \oplus S^{\lceil |\Gamma_1|/2 \rceil - (n+1) - 2} (\mathbb{C}^2))
\]

\[
\oplus (S^{n-1} (\mathbb{C}^2) \oplus S^{\lceil |\Gamma_1|/2 \rceil - (n-1) - 2} (\mathbb{C}^2))
\],

where, the first equality follows from Equation 4.4, the second equality follows from the induction hypothesis the third equality follows from the basic tensorial relation

\[
\mathbb{C}^2 \otimes S^k (\mathbb{C}^2) = S^{k-1} (\mathbb{C}^2) \oplus S^{k+1} (\mathbb{C}^2),
\]

for every \( k \geq 1 \). We observe, that either \( n - 1 \) or \( n + 1 \) are of distance \( \leq N \) from \( |\Gamma_1|/4 - 1 \) and the statement follows from the induction hypothesis.
The second statement follows from

\[ 2C(\Gamma_1)^- = C^2 \otimes C(\Gamma_1)^+ = \]
\[ = C^2 \otimes S^0(C^2) \oplus C^2 \otimes S^{[\Gamma_1]/2-1}(C^2) \]
\[ = \left(S^1(C^2) \oplus S^{[\Gamma_1]/2-1}(C^2)\right) \oplus S^{[\Gamma_1]/2-1}(C^2) \]
\[ = C(\Gamma_1)^- \oplus S^{[\Gamma_1]/2-1}(C^2), \]

where the last equality follows from the first statement, for \( n = 1 \). Note, that the second statement is an extension of the first to the case \( n = -1 \), under the convention that \( S^{-1}(C^2) = \{0\} \).

The third statement follows from

\[ S^0(C^2) \oplus 2C(\Gamma_1)^+ = S^0(C^2) \oplus C^2 \otimes C(\Gamma_1)^- = \]
\[ = S^0(C^2) \oplus C^2 \otimes S^{[\Gamma_1]/2-1}(C^2) \]
\[ = S^0(C^2) \oplus S^{[\Gamma_1]/2-2}(C^2) \oplus S^{[\Gamma_1]/2}(C^2) \]
\[ = C(\Gamma_1)^+ \oplus S^{[\Gamma_1]/2}(C^2), \]

where the last equality follows from the first statement, for \( n = 0 \).

Finally, the forth statement follows from the third statement, in case \( n = 0 \) and from the second statement, in case \( n = -1 \). Once established for two consecutive valued of \( n \), it follows for all succeeding values of \( n \), by induction, using Equation 4.4 and relation A.18.

This concludes the proof of the proposition.

References


Figure 8. The isotypic decomposition with respect of the tetrahedral subgroup of the first 12 symmetric powers.
Figure 9. Isotypic decompositions with respect to the octahedral subgroup of the first 25 symmetric powers.
Figure 10. Isotypic decomposition with respect to the icosahedral subgroup of the first 31 symmetric powers.