Bordered Floer homology via immersed curves in the punctured torus

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Outline

1. Bordered Floer homology
2. Loops and curves
3. Pairing
4. Applications
Closed manifolds:

- To a closed, orientable 3-manifold $Y$ we associate an abelian group $\widehat{HF}(Y) = H_*(\widehat{CF}(Y))$
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Manifolds with torus boundary:
- There is an algebra $\mathcal{A}$ associated to the torus.
- To an orientable 3-manifold $M$ with boundary $\partial M = T^2$ and a pair of parametrizing curves $(\alpha, \beta)$ for $\partial M$, we associate a differential module $\widehat{CFD}(M, \alpha, \beta)$ or an $A_\infty$-module $\widehat{CFA}(M, \alpha, \beta)$ over $\mathcal{A}$. 
The torus algebra $\mathcal{A}$

- $\mathcal{A}$ is generated (over $\mathbb{F}_2$) by $\rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \rho_{123}$ and two idempotents, $\iota_0$ and $\iota_1$.

$$
\rho_1 \rho_2 \rho_3 \rho_{12} \rho_{23} \rho_{123} \iota_0 \iota_1
$$
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- Multiplication is concatenation, e.g.

$$\rho_1 \rho_2 = \rho_{12}, \quad \rho_2 \rho_1 = 0, \quad \rho_1 \iota_1 = \rho_1, \quad \rho_1 \iota_0 = 0$$
The torus algebra $A$

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- $\iota_0 + \iota_1 = 1 \in A$. We also denote this by $\rho_{\emptyset}$.
A-decorated graphs

An $\mathcal{A}$ decorated graph is a directed graph with

- vertices labeled by $\iota_0$ or $\iota_1$ (we depict these labels using $\bullet$ and $\circ$, respectively)
- edges labeled by $\rho_I$ for $I \in \{1, 2, 3, 12, 23, 123, \emptyset\}$.
An $A$ decorated graph is a directed graph with

- vertices labeled by $\nu_0$ or $\nu_1$ (we depict these labels using $\bullet$ and $\circ$, respectively)
- edges labeled by $\rho_I$ for $I \in \{1, 2, 3, 12, 23, 123, \emptyset\}$.

The module $\widehat{CFD}$ (or $\widehat{CFA}$) can be represented by an $A$-decorated graph.

- vertices ↔ generators
  (each generator has an associated idempotent)
- arrows encode the differential
\( A \)-decorated graphs

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The module \( \hat{CFD} \) (or \( \hat{CFA} \)) can be represented by an \( A \)-decorated graph.

- vertices \( \leftrightarrow \) generators
  (each generator has an associated idempotent)
- arrows encode the differential

We will think of the invariants \( \hat{CFD} \) or \( \hat{CFA} \) as \( A \)-decorated graphs
  (up to appropriate equivalence)
We can always assume the graphs are \emph{reduced} (i.e. no \( \rho_\emptyset \) arrows).
Examples

\(\widehat{\text{CFD}}(D^2 \times S^1, m, \ell)\)

\(\rho_{12}\)

\(\widehat{\text{CFD}}(D^2 \times S^1, \ell, m)\)

\(\rho_{23}\)

\(\widehat{\text{CFD}}(\text{RHT}, \mu, \lambda)\)

\(\rho_1\)

\(\rho_2\)

\(\rho_3\)

\(\rho_{23}\)

\(\rho_{123}\)

\(\widehat{\text{CFD}}(\text{Fig8}, \mu, \lambda)\)

\(\rho_1\)

\(\rho_2\)

\(\rho_3\)

\(\rho_{123}\)

\(\rho_{12}\)
Loop type manifolds

At a given vertex of a reduced $\mathcal{A}$-decorated graph, we categorize the incident edges:

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Loop type manifolds

At a given vertex of a reduced \( \mathcal{A} \)-decorated graph, we categorize the incident edges:

\[ \bullet \xrightarrow{1} \bullet \xrightarrow{3} \circ \xrightarrow{2} \circ \xrightarrow{3} \]
\[ \bullet \xrightarrow{12} \bullet \xrightarrow{2} \circ \xrightarrow{23} \circ \xrightarrow{23} \]
\[ \bullet \xrightarrow{123} \bullet \xrightarrow{12} \circ \xrightarrow{1} \circ \xrightarrow{123} \]
\[ I_\bullet \quad I_{II_\bullet} \quad I_\circ \quad I_{II_\circ} \]

Definition

A loop is a connected valence two \( \mathcal{A} \)-decorated graph s.t. at every vertex, the two incident edges have types \( I_\bullet \) and \( I_{II_\bullet} \) or \( I_\circ \) and \( I_{II_\circ} \).
Loop type manifolds

At a given vertex of a reduced $\mathcal{A}$-decorated graph, we categorize the incident edges:

\[
\begin{array}{c}
\bullet \rightarrow \\
\bullet \leftarrow \\
\circ \rightarrow \\
\circ \leftarrow \\
\end{array}
\]

Definition

A *loop* is a connected valence two $\mathcal{A}$-decorated graph s.t. at every vertex, the two incident edges have types $I\circ$ and $II\circ$ or $I\bullet$ and $II\bullet$.

Definition

A 3-manifold $M$ with torus boundary is *loop type* if, up to homotopy equivalence, the graph representing $\widehat{CFD}(M, \alpha, \beta)$ is a disjoint union of loops.
Loop type manifolds

At a given vertex of a reduced $\mathcal{A}$-decorated graph, we categorize the incident edges:

\begin{align*}
\text{I} & : \bullet \quad \bullet \quad \circ \quad \circ \\
\text{II} & : \bullet \quad \circ \quad \circ \quad \circ \\
\end{align*}

Definition

A loop is a connected valence two $\mathcal{A}$-decorated graph s.t. at every vertex, the two incident edges have types I• and II• or I◦ and II◦.

Definition

A 3-manifold $M$ with torus boundary is loop type if, up to homotopy equivalence, the graph representing $\widehat{CFD}(M, \alpha, \beta)$ is a disjoint union of loops.

Note: Does not depend on the choice of parametrization $(\alpha, \beta)$.
Remark: The loop type assumption appears to be quite mild
Loop type manifolds

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- For $K \subset S^3$, if $\text{CFK}^-(K)$ admits a horizontally and vertically simplified basis, $S^3 \setminus \nu(K)$ is loop type
Loop type manifolds

**Remark:** The loop type assumption appears to be quite mild

- If $M$ has more than one L-space filling, $M$ is loop type
- For $K \subset S^3$, if $\text{CFK}^{-}(K)$ admits a horizontally and vertically simplified basis, $S^3 \setminus \nu(K)$ is loop type
- We currently do not know of any examples which are not loop type
Combinatorial description of loops

An *oriented* loop admits a well defined grading. There are four types of vertices:

\[ \text{I•, II•, II••, I••} \]

Reversing the orientation flips all the signs.

Proposition

This agrees with the relative $\mathbb{Z}_2$ grading on $\hat{CFA}$ defined by Petkova.

An oriented loop gives a cyclic word in \{•+, •−, ◦+, ◦−\}. In fact, the converse is also true.
An *oriented* loop admits a well defined grading. There are four types of vertices:

- I
- II
- I
- II

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An oriented loop gives a cyclic word in $\{\cdot, \circ, +, -\}$. In fact, the converse is also true.
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\[
\begin{array}{cccc}
& \bullet & & \\
\text{I} & & & \\
& \bullet & & \\
\text{II} & & & \\
+ & & & \\
\end{array}
\begin{array}{cccc}
& \bullet & & \\
\text{II} & & & \\
& \bullet & & \\
\text{I} & & & \\
- & & & \\
\end{array}
\begin{array}{cccc}
\circ & \circ & & \\
\text{I} & & & \\
\circ & \circ & & \\
\text{II} & & & \\
- & & & \\
\end{array}
\begin{array}{cccc}
\circ & \circ & & \\
\text{II} & & & \\
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+ & & & \\
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  & \bullet \quad \circ \\
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  & \circ \quad \bullet \\
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\bullet & \quad \bullet \\
+ & \quad - \\
\text{II} & \quad \text{I} \\
\bullet & \quad \bullet \\
- & \quad + \\
\text{I} & \quad \text{II} \\
\circ & \quad \circ \\
- & \quad - \\
\text{II} & \quad \text{I} \\
\circ & \quad \circ \\
+ & \quad +
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This agrees with the relative $\mathbb{Z}_2$ grading on $\hat{\text{CFA}}$ defined by Petkova.

An oriented loop gives a cyclic word in $\{\bullet^+, \bullet^-, \circ^+, \circ^-\}$. In fact, the converse is also true.
We will replace $\circ^\pm$ with $\alpha^\pm$ and $\bullet^\pm$ with $\beta^\pm$. We have:

oriented loops $\leftrightarrow$ cyclic words in $\alpha^\pm, \beta^\pm$
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Bordered invariants as curves

- Given a loop type manifold $M$ with parametrizing curves $\alpha$ and $\beta$, $\widehat{CFD}(M, \alpha, \beta)$ is represented by a collection of loops.
- These correspond to a collection of immersed curves in the punctured torus.
- We think of this as a collection $\gamma(M, \alpha, \beta)$ in $\partial M \setminus \{z\}$, where $z$ is a fixed basepoint.
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**Theorem 1 (H-Rasmussen-Watson)**

*The curves $\gamma(M) := \gamma(M, \alpha, \beta)$ do not depend on the parametrizing curves $\alpha$ and $\beta.*
Example: \( \mathcal{CFD}(\text{RHT}, \mu, \lambda) \)
Bordered Floer homology has a pairing theorem:

\[ \widehat{CFA}(M_1, \alpha_1, \beta_1) \boxtimes \widehat{CFD}(M_2, \alpha_2, \beta_2) \cong \widehat{CF}(M_1 \cup M_2) \]

Suppose \( M_1 \) and \( M_2 \) are loop type manifolds. Then we have collections of immersed curves \( \gamma_1 \subset \partial M_1 \) and \( \gamma_2 \subset \partial M_2 \).
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Suppose \(M_1\) and \(M_2\) are loop type manifolds. Then we have collections of immersed curves \(\gamma_1 \subset \partial M_1\) and \(\gamma_2 \subset \partial M_2\).

**Theorem 2 (H.-Rasmussen-Watson)**

Let \(Y = M_1 \cup_h M_2\), where \(h : \partial M_2 \to \partial M_1\) is a diffeomorphism. Then

\[
\widehat{HF}(Y) \cong HF(\gamma_1, h(\gamma_2)),
\]

Where right side denotes the intersection Floer homology of the two sets of curves in the punctured torus \(\partial M_1 \setminus \{z\}\).
Example

Let $Y$ be the 3-manifold obtained by splicing two RHT complements, that is, by gluing them with a map taking $\mu_1$ to $\lambda_2$ and $\lambda_1$ to $\mu_2$. 
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\[ h(\gamma_2) \]

\[ \gamma_1 \]
Application: L-space gluing

**Question:** If $M_1$ and $M_2$ are 3-manifolds with torus boundary, when is $Y = M_1 \cup M_2$ an L-space?
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Let $\mathcal{L}_{M_i}$ denote the set of L-space slopes on $\partial M_i$.

**Theorem 3 (H.-Rasmussen-Watson)**

If $M_1$ and $M_2$ are loop type and neither is the solid torus, then $M_1 \cup M_2$ is an L-space iff every slope on $\partial M_1 = \partial M_2$ is in either $\mathcal{L}_{M_1}^\circ$ or $\mathcal{L}_{M_2}^\circ$.

- If $M_1$ and $M_2$ are *simple loop type*, this was proved by H.-Watson and Rasmussen-Rasmussen.
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- This was the key remaining step in confirming a conjecture of Boyer-Gordon-Watson for graph manifolds.
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- Using curves, the proof is essentially an application of the Mean Value Theorem.
Other applications

- If $Y = M_1 \cup M_2$ is a toroidal integer homology sphere and both sides are loop type, $Y$ is not an L-space.
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- Rank inequality for pinching
  \[ \text{rk } \widehat{\text{HF}}(M_1 \cup M_2) \geq \text{rk } \widehat{\text{HF}}(M_1 \cup D^2 \times S^1) \]
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- Connections to Seiberg-Witten theory?
- Recovering $HF^+$?
Thank you!