9 Riemannian metrics

Differential forms and the exterior derivative provide one piece of analysis on manifolds which, as we have seen, links in with global topological questions. There is much more one can do when one introduces a Riemannian metric. Since the whole subject of Riemannian geometry is a huge one, we shall here look at only two aspects which relate to the use of differential forms: the study of harmonic forms and of geodesics. In particular, we ignore completely here questions related to curvature.

9.1 The metric tensor

In informal terms, a Riemannian metric on a manifold M is a smoothly varying positive definite inner product on the tangent spaces T_x. To make global sense of this, note that an inner product is a bilinear form, so at each point x we want a vector in the tensor product

$$T_x^* \otimes T_x^*.$$

We can put, just as we did for the exterior forms, a vector bundle structure on

$$T^*M \otimes T^*M = \bigcup_{x \in M} T_x^* \otimes T_x^*.$$

The conditions we need to satisfy for a vector bundle are provided by two facts we used for the bundle of p-forms:

- each coordinate system $x_1, \ldots, x_n$ defines a basis $dx_1, \ldots, dx_n$ for each T_x^* in the coordinate neighbourhood and the n^2 elements

  $$dx_i \otimes dx_j, \quad 1 \leq i, j \leq n$$

  give a corresponding basis for $T_x^* \otimes T_x^*$
- the Jacobian of a change of coordinates defines an invertible linear map $J : T_x^* \to T_x^*$ and we have a corresponding invertible map $J \otimes J : T_x^* \otimes T_x^* \to T_x^* \otimes T_x^*$.

Given this, we define:

Definition 30 *A Riemannian metric on a manifold M is a section g of $T^* \otimes T^*$ which at each point is symmetric and positive definite.*

In a local coordinate system we can write

$$g = \sum_{i,j} g_{ij}(x) dx_i \otimes dx_j$$

where $g_{ij}(x) = g_{ji}(x)$ and is a smooth function, with $g_{ij}(x)$ positive definite. Often the tensor product symbol is omitted and one simply writes

$$g = \sum_{i,j} g_{ij}(x) dx_i dx_j.$$

Example:

1. The Euclidean metric on $\mathbf{R}^n$ is defined by

$$g = \sum dx_i \otimes dx_i.$$

So

$$g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \delta_{ij}.$$

2. A submanifold of $\mathbf{R}^n$ has an induced Riemannian metric: the tangent space at x can be thought of as a subspace of $\mathbf{R}^n$ and we take the Euclidean inner product on $\mathbf{R}^n$.

Given a smooth map $F : M \to N$ and a metric g on N, we can pull back g to a section F^*g of $T^*M \otimes T^*M$:

$$(F^*g)_x(X, Y) = g_F(x)(DF_x(X), DF_x(Y)).$$

If DF_x is invertible, this will again be positive definite, so in particular if F is a diffeomorphism.

Definition 31 *A diffeomorphism $F : M \to N$ between two Riemannian manifolds is an isometry if $F^*g_N = g_M$.*

Example: Let $M = \{(x, y) \in \mathbf{R}^2 : y > 0\}$ and

$$g = \frac{dx^2 + dy^2}{y^2}.$$

If $z = x + iy$ and

$$F(z) = \frac{az + b}{cz + d}$$

with a, b, c, d real and $ad - bc > 0$, then

$$F^* dz = (ad - bc)\frac{dz}{(cz + d)^2}$$

and

$$F^* y = y \circ F = \frac{1}{i}\left(\frac{az + b}{cz + d} - \frac{a\bar{z} + b}{c\bar{z} + d}\right) = \frac{ad - bc}{|cz + d|^2} y.$$

Then

$$F^* g = (ad - bc)^2 \frac{dx^2 + dy^2}{|(cz + d)^2|^2} \frac{|cz + d|^4}{(ad - bc)^2 y^2} = \frac{dx^2 + dy^2}{y^2} = g.$$

So these Möbius transformations are isometries of a Riemannian metric on the upper half-plane.

This is the non-Euclidean geometry dealt with in the Projective Geometry Notes.

With a Riemannian metric one can define the length of a curve:

Definition 32 *Let M be a Riemannian manifold and $\gamma : [0, 1] \to M$ a smooth map (i.e. a smooth curve in M). The length of the curve is*

$$\ell(\gamma) = \int_0^1 \sqrt{g(\gamma', \gamma')} dt$$

where $\gamma'(t) = D\gamma_t(d/dt)$.

With this definition, any Riemannian manifold is a metric space: define

$$d(x, y) = \inf\{\ell(\gamma) \in \mathbf{R} : \gamma(0) = x, \gamma(1) = y\}.$$

Are Riemannian manifolds special? No, because:

Proposition 9.1 *Any manifold admits a Riemannian metric.*

Proof: Take a covering by coordinate neighbourhoods and a partition of unity subordinate to the covering. On each open set U_α we have a metric

$$g_\alpha = \sum_i dx_i^2$$

in the local coordinates. Define

$$g = \sum \varphi_i g_{\alpha(i)}.$$

This sum is well-defined because the supports of φ_i are locally finite. Since $\varphi_i \geq 0$ at each point every term in the sum is positive definite or zero, but at least one is positive definite so the sum is positive definite. □

9.2 The geodesic flow

Consider any manifold M and its cotangent bundle T^*M, with projection $p : T^*M \to M$. Let X be tangent vector to T^*M at the point $\xi_a \in T_a^*$. Then

$$Dp_{\xi_a}(X) \in T_aM$$

so

$$\theta(X) = \xi_a(Dp_{\xi_a}(X))$$

defines a canonical 1-form θ on T^*M. In coordinates $(x, y) \mapsto \sum_i y_i dx_i$, the projection p is

$$p(x, y) = x$$

so if

$$X = \sum a_i \frac{\partial}{\partial x_i} + \sum b_i \frac{\partial}{\partial y_i}$$

then

$$\theta(X) = \sum_i y_i dx_i(Dp_{\xi_a}X) = \sum_i y_i a_i$$

which gives

$$\theta = \sum_i y_i dx_i.$$

We now take the exterior derivative

$$\omega = -d\theta = \sum dx_i \wedge dy_i$$

which is the *canonical* 2-*form* on the cotangent bundle. It is non-degenerate, so that the map

$$X \mapsto i(X)\omega$$

from the tangent bundle of T^*M to its cotangent bundle is an isomorphism.

Now suppose f is a smooth function on T^*M. Its derivative is a 1-form df. Because of the isomorphism above, there is a unique vector field X on T^*M such that

$$i(X)\omega = df.$$

If g is another function with vector field Y, then

$$Y(f) = df(Y) = i(Y)i(X)\omega = -i(X)i(Y)\omega = -X(g) \tag{20}$$

On a Riemannian manifold we shall see next that there is a natural function on T^*M. In fact a metric defines an inner product on T^* as well as on T, for the map

$$X \mapsto g(X, -)$$

defines an isomorphism from T to T^*. In concrete terms, if g^* is the inner product on T^*, then

$$g^*(\sum_j g_{ij} dx_j, \sum_k g_{kl} dx_l) = g_{ik}$$

which means that

$$g^*(dx_j, dx_k) = g^{jk}$$

where g^{jk} denotes the inverse matrix to g_{jk}.

We consider the function on T^*M defined by

$$H(\xi_a) = g^*(\xi_a, \xi_a).$$

In local coordinates this is

$$H(x, y) = \sum_{ij} g^{ij}(x) y_i y_j.$$

Definition 33 *The vector field X on T^*M given by $i(X)\omega = dH$ is called the geodesic flow of the metric g.*

Definition 34 *If $\gamma : (a, b) \to T^*M$ is an integral curve of the geodesic flow, then the curve $p(\gamma)$ in M is called a geodesic.*

In local coordinates, if the geodesic flow is

$$X = \sum a_i \frac{\partial}{\partial x_i} + \sum b_i \frac{\partial}{\partial y_i}$$

then

$$i(X)\omega = \sum_k (a_k dy_k - b_k dx_k) = dH = \sum_{ij} \frac{\partial g^{ij}}{\partial x_k} dx_k y_i y_j + 2 \sum_{ij} g^{ij} y_i dy_j.$$

Thus the integral curves are solutions of

$$\frac{dx_k}{dt} = 2 \sum_j g^{kj} y_j \tag{21}$$

$$\frac{dy_k}{dt} = -\sum_{ij} \frac{\partial g^{ij}}{\partial x_k} y_i y_j \tag{22}$$

Before we explain why this is a geodesic, just note the qualitative behaviour of these curves. For each point $a \in M$, choose a point $\xi_a \in T_a^*$ and consider the unique integral curve starting at ξ_a. Equation (21) tells us that the projection of the integral curve is parallel at a to the tangent vector X_a such that $g(X_a, -) = \xi_a$. Thus these curves have the property that through each point and in each direction there passes one geodesic.

Geodesics are normally thought of as curves of shortest length, so next we shall link up this idea with the definition above. Consider the variational problem of looking for critical points of the length functional

$$\ell(\gamma) = \int_0^1 \sqrt{g(\gamma', \gamma')} dt$$

for curves with fixed end-points $\gamma(0) = a, \gamma(1) = b$. For simplicity assume a, b are in the same coordinate neighbourhood. If

$$F(x, z) = \sum_{ij} g_{ij}(x) z_i z_j$$

then the first variation of the length is

$$\begin{aligned} \delta\ell &= \int_0^1 \frac{1}{2} F^{-1/2} \left(\frac{\partial F}{\partial x_i} \dot{x}_i + \frac{\partial F}{\partial z_i} \frac{d\dot{x}_i}{dt} \right) dt \\ &= \int_0^1 \frac{1}{2} F^{-1/2} \frac{\partial F}{\partial x_i} \dot{x}_i - \frac{d}{dt} \left(\frac{1}{2} F^{-1/2} \frac{\partial F}{\partial z_i} \right) \dot{x}_i dt. \end{aligned}$$

on integrating by parts with $\dot{x}_i(0) = \dot{x}_i(1) = 0$. Thus a critical point of the functional is given by

$$\frac{1}{2} F^{-1/2} \frac{\partial F}{\partial x_i} - \frac{d}{dt} \left(\frac{1}{2} F^{-1/2} \frac{\partial F}{\partial z_i} \right) = 0$$

If we parametrize this critical curve by arc length:

$$s = \int_0^t \sqrt{g(\gamma', \gamma')} dt$$

then $F = 1$, and the equation simplifies to

$$\frac{\partial F}{\partial x_i} - \frac{d}{dt} \left(\frac{\partial F}{\partial z_i} \right) = 0.$$

But this is

$$\sum \frac{\partial g_{jk}}{\partial x_i} \frac{dx_j}{dt} \frac{dx_k}{dt} - \frac{d}{dt} \left(2 g_{ik} \frac{dx_k}{dt} \right) = 0 \tag{23}$$

But now define y_i by

$$\frac{dx_k}{dt} = 2\sum_j g^{kj} y_j$$

as in the first equation for the geodesic flow (21) and substitute in (23) and we get

$$4\sum \frac{\partial g_{jk}}{\partial x_i} g^{ja} y_a g^{kb} y_b - \frac{d}{dt}\left(4 g_{ik} g^{ka} y_a\right) = 0$$

and using

$$\sum_j g^{ij} g_{jk} = \delta^i_k$$

this yields

$$-\frac{\partial g^{jk}}{\partial x_i} y_j y_k = \frac{dy_i}{dt}$$

which is the second equation for the geodesic flow. (Here we have used the formula for the derivative of the inverse of a matrix G: $D(G^{-1}) = -G^{-1}DGG^{-1}$).

The formalism above helps to solve the geodesic equations when there are isometries of the metric. If $F : M \to M$ is a diffeomorphism of M then its natural action on 1-forms induces a diffeomorphism of T^*M. Similarly with a one-parameter group φ_t. Differentiating at $t = 0$ this means that a vector field X on M induces a vector field $\tilde{X}$ on T^*M. Moreover, the 1-form θ on T^*M is canonically defined and hence invariant under the induced action of any diffeomorphism. This means that

$$\mathcal{L}_{\tilde{X}}\theta = 0$$

and therefore, using (6.5) that

$$i(\tilde{X})d\theta + d(i(\tilde{X})\theta) = 0$$

so since $\omega = -d\theta$

$$i(\tilde{X})\omega = df$$

where $f = i(\tilde{X})\theta$.

Proposition 9.2 *Any vector field Y on T^*M for which $\mathcal{L}_Y\theta = 0$ is the vector field $\tilde{X}$ induced from a vector field X on M. The function $f = i(\tilde{X})\theta$ is $f(\xi_x) = \xi_x(X_x)$.*

Proof: Write in coordinates

$$Y = \sum a_i \frac{\partial}{\partial x_i} + \sum b_i \frac{\partial}{\partial y_i}$$

where $\theta = \sum_i y_i dx_i$ then $\mathcal{L}_Y\theta = 0$ gives

$$0 = \sum_i b_i dx_i + \sum_{i,j} y_i(\frac{\partial a_i}{\partial x_j}dx_j + \frac{\partial a_i}{\partial y_j}dy_j).$$

This implies that a_i is independent of y_i and so

$$X = \sum_i a_i \frac{\partial}{\partial x_i}$$

is the required vector field X on M. We have

$$i(\tilde{X})\theta = \sum_i a_i(x)y_i = \xi_x(X_x)$$

by the definition of θ. □

Now let M be a Riemannian manifold and H the function on T^*M defined by the metric as above. If φ_t is a one-parameter group of *isometries*, then the induced diffeomorphisms of T^*M will preserve the function H and so the vector field $\tilde{Y}$ will satisfy

$$\tilde{Y}(H) = 0.$$

But from (20) this means that $X(f) = 0$ where X is the geodesic flow and f the function $i(\tilde{Y})\theta$. This function is constant along the geodesic flow, and is therefore a constant of integration of the geodesic equations.

Example: Consider the metric

$$g = \frac{dx_1^2 + dx_2^2}{x_2^2}$$

on the upper half plane and its geodesic flow X.

The map $(x_1, x_2) \mapsto (x_1 + t, x_2)$ is clearly a one-parameter group of isometries (the Möbius transformations $z \mapsto z + t$) and defines the vector field

$$Y = \frac{\partial}{\partial x_1}.$$

On the cotangent bundle this gives the function

$$f(x, y) = y_1$$

which is constant on the integral curve.

The map $z \mapsto e^t z$ is also an isometry with vector field

$$Z = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$$

so that

$$g(x, y) = x_1 y_1 + x_2 y_2$$

is constant.

We also have automatically that $H = x_2^2(y_1^2 + y_2^2)$ is constant since

$$X(H) = i(X)i(X)\omega = 0.$$

We therefore have three equations for the integral curves of the geodesic flow:

$$\begin{aligned} y_1 &= c_1 \\ x_1 y_1 + x_2 y_2 &= c_2 \\ x_2^2(y_1^2 + y_2^2) &= c_3 \end{aligned}$$

Eliminating y_1, y_2 gives the geodesics:

$$(c_1 x_1 - c_2)^2 + c_1^2 x_2^2 = c_3.$$

If $c_1 = 0$ this is a half-line $x_2 = const.$. Otherwise it is a semicircle with centre on the x_1 axis. These are the straight lines of non-Euclidean geometry as described in the Projective Geometry notes.

9.3 Harmonic forms

We mentioned above that a metric g defines an inner product not just on T_a but also an inner product g^* on T_a^*. With this we can define an inner product on the pth exterior power $\Lambda^p T_a^*$:

$$(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_p, \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_p) = \det g^*(\alpha_i, \beta_j) \tag{24}$$

In particular, on an n-manifold there is an inner product on each fibre of the bundle $\Lambda^n T^*$. Since each fibre is one-dimensional there are only two unit vectors $\pm u$.

Definition 35 *Let M be an oriented Riemannian manifold, then the volume form is the unique n-form ω of unit length in the equivalence class defined by the orientation.*

In local coordinates,the definition of the inner product (24) gives

$$(dx_1 \wedge \ldots \wedge dx_n, dx_1 \wedge \ldots \wedge dx_n) = \det g^*_{ij} = (\det g_{ij})^{-1}$$

Thus if $dx_1 \wedge \ldots \wedge dx_n$ defines the orientation,

$$\omega = \sqrt{\det g_{ij}} dx_1 \wedge \ldots \wedge dx_n.$$

On a compact manifold we can integrate this to obtain the total volume – so a metric defines not only lengths but also volumes.

Now take $\alpha \in \Lambda^p T^*_a, \beta \in \Lambda^{n-p} T^*_a$ and define $f_\beta : \Lambda^p T^*_a \to \mathbf{R}$ by

$$f_\beta(\alpha)\omega = \beta \wedge \alpha.$$

But we have an inner product, so any linear map on $\Lambda^p T^*_a$ is of the form

$$\alpha \mapsto (\alpha, \gamma)$$

for some $\gamma \in \Lambda^p T^*_a$, so we have a well-defined linear map $\beta \mapsto \gamma_\beta$ from $\Lambda^{n-p} T^*$ to $\Lambda^p T^*$, satisfying

$$(\gamma_\beta, \alpha)\omega = \beta \wedge \alpha.$$

We use a different symbol for this:

Definition 36 *The Hodge star operator is the linear map* $* : \Omega^p(M) \to \Omega^{n-p}(M)$ *with the property that at each point*

$$(\alpha, \beta)\omega = \alpha \wedge *\beta.$$

Example: If $e_1, \ldots, e_n$ is an orthonormal basis of the space of one-forms at a point, then

$$*(e_1 \wedge \ldots \wedge e_p) = e_{p+1} \wedge \ldots \wedge e_n.$$

Exercise 9.3 *Show that on p-forms,* $*^2 = (-1)^{p(n-p)}$.

On a Riemannian manifold we can use the star operator to define new differential operators on forms. In particular, consider the operator

$$d^* : \Omega^p(M) \to \Omega^{p-1}(M)$$

defined by

$$d^* = (-1)^{np+n+1} * d * .$$

The notation is suggestive, in fact:

Proposition 9.4 *Let M be an oriented Riemannian manifold with volume form ω and let $\alpha \in \Omega^p(M), \beta \in \Omega^{p-1}(M)$ be forms of compact support. Then*

$$\int_M (d^*\alpha, \beta)\omega = \int_M (\alpha, d\beta)\omega.$$

Proof: We have

$$\int_M (d^*\alpha, \beta)\omega = (-1)^{np+n+1} \int_M (*d{*}\alpha, \beta)\omega = (-1)^{np+n+1} \int_M (\beta, *d{*}\alpha)\omega = (-1)^{np+n+1} \int_M \beta \wedge **d{*}\alpha$$

from the definition of d^* and $*$. But on the $n-p+1$-form $d*\alpha$, $*^2 = (-1)^{(n-p+1)(p-1)}$ so this is

$$(-1)^{np+n+1+(n-p+1)(p-1)} \int_M \beta \wedge d * \alpha = (-1)^p \int_M \beta \wedge d * \alpha.$$

Now

$$d(\beta \wedge *\alpha) = d\beta \wedge *\alpha + (-1)^{p-1}\beta \wedge d * \alpha.$$

Integrating $d(\beta \wedge *\alpha)$ gives zero from the first version of Stokes' theorem (7.2), so we get

$$(-1)^p \int_M \beta \wedge d * \alpha = \int_M d\beta \wedge *\alpha = \int_M (\alpha, d\beta)\omega.$$

□

Definition 37 *Let M be an oriented Riemannian manifold, then the Laplacian on p-forms is the differential operator $\Delta : \Omega^p(M) \to \Omega^p(M)$ defined by*

$$\Delta = dd^* + d^*d.$$

Example: Suppose $M = \mathbf{R}^3$ with the Euclidean metric and $\alpha = a_1 dx_1$, then

$$\Delta(a_1 dx_1) = (dd^* + d^*d)(a_1 dx_1)$$

so

$$dd^*(a_1 dx_1) = -d * d * (a_1 dx_1) = -d * d(a_1 dx_2 \wedge dx_3) = -d * \frac{\partial a_1}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3$$

$$= -d\frac{\partial a_1}{\partial x_1} = -\frac{\partial^2 a_1}{\partial x_1^2}dx_1 - \frac{\partial^2 a_1}{\partial x_2 \partial x_1}dx_2 - \frac{\partial^2 a_1}{\partial x_3 \partial x_1}dx_3$$

and

$$d^*d(a_1dx_1) = d^*(\frac{\partial a_1}{\partial x_2}dx_2 \wedge dx_1 + \frac{\partial a_1}{\partial x_3}dx_3 \wedge dx_1) = *d(\frac{\partial a_1}{\partial x_2}dx_3 - \frac{\partial a_1}{\partial x_3}dx_2)$$

$$= *(\frac{\partial^2 a_1}{\partial x_1 \partial x_2}dx_1 \wedge dx_3 + \frac{\partial^2 a_1}{\partial x_2^2}dx_2 \wedge dx_3 - \frac{\partial^2 a_1}{\partial x_1 \partial x_3}dx_1 \wedge dx_2 - \frac{\partial^2 a_1}{\partial x_3^2}dx_3 \wedge dx_2)$$

$$= \frac{\partial^2 a_1}{\partial x_1 \partial x_2}dx_2 - \frac{\partial^2 a_1}{\partial x_2^2}dx_1 + \frac{\partial^2 a_1}{\partial x_1 \partial x_3}dx_3 - \frac{\partial^2 a_1}{\partial x_3^2}dx_1.$$

Adding, we get

$$\Delta(a_1dx_1) = -\left(\frac{\partial^2 a_1}{\partial x_1^2} + \frac{\partial^2 a_1}{\partial x_2^2} + \frac{\partial^2 a_1}{\partial x_3^2}\right)dx_1$$

which is the negative of the usual Laplacian on the coefficient a_1. By linearity the same is true for a general 1-form $a_1dx_1 + a_2dx_2 + a_3dx_3$.

When $p = 0$ we have

$$\Delta f = d^*df = (-1)^{n+n+1} * d * df = - * d * df$$

and this is sometimes called the *Laplace-Beltrami operator*, though there are differing conventions about sign:

Example:

1. Take $M = \mathbf{R}^n$ with the Euclidean metric.

$$\begin{aligned}
df &= \sum_i \frac{\partial f}{\partial x_i}dx_i \\
*df &= \frac{\partial f}{\partial x_1}dx_2 \wedge \ldots \wedge dx_n + \ldots \\
d * df &= \frac{\partial^2 f}{\partial x_1^2}dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n + \ldots \\
\Delta f = - * d * df &= -\sum_i \frac{\partial^2 f}{\partial x_i^2}
\end{aligned}$$

2. Take M to be the upper half-plane with metric

$$g = \frac{1}{y^2}(dx^2 + dy^2).$$

Then

$$\omega = \frac{1}{y^2} dx \wedge dy \qquad * dx = dy, \quad *dy = -dx.$$

So

$$\begin{aligned}\Delta f &= -*d(\frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx) \\ &= -y^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right)\end{aligned}$$

It follows that the real and imaginary part of a holomorphic function of $z = x + iy$ satisfy $\Delta f = 0$, just as in the case of the Euclidean metric.

Definition 38 *A differential form $\alpha \in \Omega^p(M)$ is a harmonic form if $\Delta\alpha = 0$.*

On a compact manifold harmonic forms play a very important role, which there is no time to explore in this course. Here is the starting point:

Proposition 9.5 *Let M be a compact oriented Riemannian manifold. Then*

- *a p-form is harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$*
- *in each de Rham cohomology class there is at most one harmonic form.*

Proof: Clearly if $d\alpha = d^*\alpha = 0$, then $(dd^* + d^*d)\alpha = 0$. Suppose conversely $\Delta\alpha = 0$, then

$$0 = \int_M (\Delta\alpha, \alpha)\omega = \int_M (dd^* + d^*d)\alpha, \alpha)\omega = \int_M (d^*\alpha, d^*\alpha)\omega + \int_M (d\alpha, d\alpha)\omega.$$

But these last two terms are non-negative and vanish if and only if $d\alpha = d^*\alpha = 0$.

Suppose α, α' are harmonic forms in the same cohomology class, then

$$\alpha - \alpha' = d\beta.$$

But then

$$0 = d^*\alpha - d^*\alpha' = d^*d\beta$$

and

$$0 = \int_M (d^*d\beta, \beta)\omega = \int_M (d\beta, d\beta)\omega$$

which gives $d\beta = 0$ and $\alpha = \alpha'$. □

The theorem of W.V.D. Hodge says that there exists in each cohomology class a harmonic form, which as we have seen is unique. This result was a profound influence on geometry in the last half of the 20th century. The proof is far beyond the scope of this course, but the interested reader with a week or two to spare can find a proof in: *Foundations of Differentiable manifolds and Lie Groups* by F. Warner, Graduate Texts in Mathematics **94**, Springer 1983. There is a natural interpretation of the result: the harmonic form α in a cohomology class is the one of smallest $\mathcal{L}^2$ norm, because any other is of the form $\alpha + d\beta$ and

$$\int_M (\alpha + d\beta, \alpha + d\beta)\omega = \int_M (\alpha, \alpha)\omega + \int_M (d\beta, d\beta)\omega \geq \int_M (\alpha, \alpha)\omega$$

since

$$\int_M (\alpha, d\beta)\omega = \int_M (d^*\alpha, \beta)\omega = 0.$$

There are some immediate consequences of the Hodge theorem. First note that:

Proposition 9.6 *The Laplacian Δ commutes with $*$.*

Proof:

$$\begin{aligned}(dd^* + d^*d) * \alpha &= (-1)^{n(n-p)+n+1} d * d * * \alpha + (-1)^{n(n-p+1)+n+1} * d * d * \alpha \\ &= (-1)^{n(n-p)+n+1+p(n-p)} d * d\alpha + (-1)^{n+pn+1} * d * d * \alpha \\ &= (-1)^{p+1} d * d\alpha + (-1)^{n+pn+1} * d * d * \alpha\end{aligned}$$

and

$$\begin{aligned}*(dd^* + d^*d)\alpha &= (-1)^{np+n+1} * d * d * \alpha + (-1)^{n(p+1)+n+1} d * d * * \alpha \\ &= (-1)^{np+n+1} * d * d * \alpha + (-1)^{n(p+1)+n+p(n-p)+1} d * d\alpha \\ &= (-1)^{np+n+1} * d * d * \alpha + (-1)^{p+1} d * d\alpha\end{aligned}$$

□

It follows from the proposition that $*$ maps harmonic forms to harmonic forms. since $*^2 = (-1)^{p(n-p)}$ it is invertible and so it maps the space of harmonic p-forms isomorphically to the space of harmonic $n-p$ forms. One consequence of the Hodge theorem is that

$$\dim H^p(M) = \dim H^{n-p}(M).$$

This we saw for $p = 0$ rather differently in Theorem 8.2.