

Chapter 2

General Topology

In this chapter, we will start with the real number line and investigate some of its properties. We will then define a topological space as an abstraction of features of the real line. The topological ideas of limit point, convergence, open and closed sets, and continuity are all the result of capturing essential characteristics that we find in the real numbers.

2.1 The Real Number Line

We will not present an axiomatic definition of the real numbers. Instead, we will rely on our understanding of the real number line as the set of all decimal numbers ordered in their familiar way.

Let us first review the concepts necessary to define convergence of sequences and continuity of functions on the real number line.

Definition (open interval). In the real number line \mathbb{R} define an *open interval* (a, b) as the set $\{x \in \mathbb{R} \mid a < x < b\}$.

Definition (open interval centered at x). Given $x \in \mathbb{R}$ and $\varepsilon > 0$ the *open interval centered at x of radius ε* , $B(x, \varepsilon)$ is the open interval $(x - \varepsilon, x + \varepsilon)$.

Definition (open set in \mathbb{R}). In \mathbb{R} a set U is *open* if and only if for every point $x \in U$ there is an $\varepsilon_x > 0$ such that $(x - \varepsilon_x, x + \varepsilon_x) \subseteq U$.

Theorem 2.1. The empty set is open and \mathbb{R} is open.

Theorem 2.2. If U_1 and U_2 are open sets, then $U_1 \cap U_2$ is open. In fact, the intersection of finitely many open sets is open.

Theorem 2.3. The union of any collection of open sets in \mathbb{R} is open.

Theorem 2.4. If U is open, then U is the union of open intervals.

Let us recall the definition of convergence of a sequence in \mathbb{R} :

Definition (convergent sequence). We say a sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq \mathbb{R}$ *converges* to x , or that x is the *limit of the sequence*, written as $x_i \rightarrow x$ if and only if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $|x - x_i| < \varepsilon$ for all $i > N$.

We are in the process of recasting ideas from the real line in set-theoretic terms. So let's rephrase the definition of convergence *without* using distance, but, instead, in terms of open intervals. The definition above says that any open interval of radius ε centered at x contains *all but finitely many* of the elements of the sequence. Instead of restricting ourselves to open intervals centered at x , we can consider *any* open set containing x and define convergence as follows:

Definition (convergent sequence). If $\{x_i\}_{i \in \mathbb{N}} \subseteq \mathbb{R}$, then we say that x *converges* to x , written as $x_i \rightarrow x$ if and only if for any open set containing x , there is an $N \in \mathbb{N}$ such that $x_i \in U$ for all $i > N$.

If instead of studying sequences we wish to look at sets in \mathbb{R} in general, we would like to define what we mean by saying that a point is “close” to (other) points in that set. One way is to abstract the concept of distance, but we don't actually need to have a distance, we can also use the (more general) concept of an open set. Let us see how it would play out in \mathbb{R} .

We could say that x is “close” to a set A if there is a sequence of elements of A that converges to x . If x is in A , however, we could always cheat and pick the constant sequence $\{x_i\}$ where $x_i = x$ for all (or almost all) i . This definition would not include the idea that x has nearby points from A . To avoid this problem we could say that x is “close” to a set A if there is a sequence $\{x_i\}$ of elements of $A - \{x\}$ that converges to x . But a more general definition would be to dispense with sequences altogether, and use the condition that the intersection of any open set containing x with $A - \{x\}$ is never empty. So we thus come to the following definition:

Definition (limit point in \mathbb{R}). Let A be a subset of \mathbb{R} and x be a point in \mathbb{R} . Then x is a *limit point* of A if and only if for every open set U containing x $(U - \{x\}) \cap A$ is not empty.

In other words, a limit point of a set is one that cannot be isolated from the rest of the set with an open set.

Definition (closed set in \mathbb{R}). A set in \mathbb{R} is *closed* if and only if it contains all of its limit points.

Theorem 2.5. The intersection of any collection of closed sets in \mathbb{R} is closed.

Theorem 2.6. The union of two closed sets in \mathbb{R} is closed. In fact, the union of finitely many closed sets in \mathbb{R} is closed.

Closed sets and open sets are related by the following theorem.

Theorem 2.7. A set in \mathbb{R} is open if and only if its complement is closed.

Let us now review the definition of continuity that you probably first encountered in calculus—the ε - δ definition.

Definition (continuous function in \mathbb{R}). A function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is *continuous at x* if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $z \in D$ and $|x - z| < \delta$ then $|f(x) - f(z)| < \varepsilon$. We say f is *continuous* if it is continuous for every point x in its domain D .

We wish to convert this definition into the language of open sets and intervals. It is saying that if a function is continuous in its domain, then if we pick an open interval $I = (f(x) - \varepsilon, f(x) + \varepsilon)$, then we can find an open interval $J = (x - \delta, x + \delta)$ that is mapped into I . So, in general, if we pick an open set $U \subseteq \mathbb{R}$ that contains a point $f(x)$, then we can find an open set $V \subseteq \mathbb{R}$ containing x whose image is contained inside U .

This set theoretic view of continuity allows us to re-word the concept of continuity in the language of open sets as follows:

Definition (continuous function in \mathbb{R}). A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a *continuous function* if and only if for every open set U in \mathbb{R} , $f^{-1}(U)$ is open in \mathbb{R} .

Theorem 2.8. The two definitions of continuity for real-valued functions on \mathbb{R} are equivalent.