Harmonic Maps of Riemann Surfaces

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1 Introduction

Variational methods play an important role in mathematical physics and differential geometry, optimal control and numerical analysis. Many problems of functional analysis can be cast into the form of functional equations $F(u) = 0$, the solution u being sought among a class of admissible functions. A particular class of functional equations is the class of Euler-Lagrange equations

$$DE(u) = 0$$

for a Fréchet differentiable functional E with derivative DE on a Banach space V. For such E we call a point $u \in V$ *critical* if $DE(u) = 0$. Of particular interest are the relative minima of E, possibly subject to constraints.

In general, in order to achieve a satisfactory existence theory the notion of solution must be suitably relaxed. Hence, this method will at first only yield "weak" solutions to the problems. A second step is necessary to show that these solutions are regular enough to be admitted as classical solutions. The regularity theory in many cases is very subtle, but it is relatively trivial in the particular case of harmonic maps.

We recall the basic result that will be needed:

Theorem 1.1. *Suppose V is a reflexive Banach space with norm $||\cdot||$, and let $M \subset V$ be a weakly closed subset of V. Suppose $E : M \to \mathbb{R} \cup +\infty$ is coercive and (sequentially) weakly lower semi-continuous on M with respect to V, that is, suppose the following conditions are fulfilled:*

(1°) $E(u) \to \infty$ as $||u|| \to \infty$, $u \in M$.

(2°)For any $u \in M$, any sequence u_m in M such that $u_m \rightharpoonup u$ weakly in V there holds:

$$E(u) \le \liminf_{m\to\infty} E(u_m).$$

Then E is bounded from below on M and attains its infimum on M.

Given a smooth compact Riemann surface (one-dimensional complex manifold) Σ with metric γ, for simplicity without boundary, and any smooth compact k-dimensional manifold N with metric g, a natural generalization of the Dirichlet's integral for C^1-functions on a domain in $\mathbb{R}^n$ is the following energy functional

$$E(u) = \int_\Sigma e(u) d\Sigma.$$

Here in local coordinates on Σ and N the energy density $e(u)$ is given by

$$e(u) = \sum_{1\le\alpha,\beta\le 2} \sum_{1\le i,j\le n} \frac{1}{2}\gamma^{\alpha\beta}(x) g_{ij}(u) \frac{\partial}{\partial x_\alpha} u^i \frac{\partial}{\partial x_\beta} u^j,$$

with $\gamma^{\alpha\beta} = (\gamma_{\alpha\beta})^{-1}$ denoting the coefficients of the inverse matrix $(\gamma_{\alpha\beta})$ representing the metric γ, (g_{ij}) representing g, and with

$$d\Sigma = \sqrt{|\gamma|} dx, \ |\gamma| = \det(\gamma_{\alpha\beta}).$$

Since we assume that both Σ and N are compact, this expression may be simplified considerably: First, by the Nash embedding theorem, see for instance [N] or [Sc; pp. 43-45], any compact Riemannian manifold N may be isometrically embedded into some Euclidean space $\mathbb{R}^n$. A new proof that avoids the hard implicit function theorem has recently been obtained by Günther [G].

Moreover, E is invariant under conformal mappings of Σ; this means that if $f : \Gamma \to \Sigma$ is a bijective holomorphic mapping between surfaces, then $E(u \circ f) = E(u)$. Thus, by the uniformization theorem (that says that any surface is conformally equivalent to a surface with constant curvature) and the classification of surfaces we may assume that either $\Sigma = S^2$ or $\Sigma = T^2 = \mathbb{R}^2/\mathbb{Z}^2$, or is a quotient of the upper half-space $\mathbb{H}$, endowed with the hyperbolic metric $\frac{1}{y} dx\, dy$. In particular, if $\Sigma = T^2$, the energy density is simply given by $e(u) = \frac{1}{2}|\nabla u|^2$ and E becomes the standard Dirichlet integral for mappings $u : T^2 = \mathbb{R}^2/\mathbb{Z}^2 \to N \subset \mathbb{R}^n$.

Consider the space $C^1(\Sigma; N)$ of C^1-functions $u : \Sigma \to N \hookrightarrow \mathbb{R}^n$, then $C^1(\Sigma; N)$ is an infinite-dimensional manifold with tangent space given by

$$T_u C^1(\Sigma; N) = \{\varphi \in C^1(\Sigma; \mathbb{R}^n) \ ; \ \varphi(x) \in T_{u(x)}N \quad \text{for} \quad x \in \Sigma\},$$

and the functional E is differentiable on this space.

In fact, if we consider variations $\varphi \in T_u C^1(\Sigma; N)$ such that $\operatorname{supp}(\varphi)$ is contained in a chart U on Σ whose image $u(U)$ is contained in a coordinate patch of N, then in order to compute the variation of E in direction φ it suffices to work in one coordinate frame – both on Σ and N – and all computations can be done as in the "flat" case. From this, the differentiability of E is immediate and the following definition is meaningful.

Definition 1.2. *A stationary point $u \in C^1(\Sigma, N)$ of E is called a harmonic map.*

The concept of harmonic map generalizes the notion of (closed) geodesic to higher dimensions. Moreover, if we choose $N = \mathbb{R}^n$, we see that harmonic functions simply appear as special cases of harmonic maps. In order to become familiar with this new notion we derive the Euler-Lagrange equation for harmonic maps.

2 The Euler-Lagrange equation for Harmonic Maps

As a model case we first consider the case $\Sigma = T^2 = \mathbb{R}^2/\mathbb{Z}^2$, where E reduces to the standard Dirichlet integral

$$E(u) = \int_{[0,1]\times[0,1]} |\nabla u|^2 dx$$

for doubly periodic mappings $u : \mathbb{R}^2 \to N \subset \mathbb{R}^n$.

In this case, if $u : T^2 \to N \subset \mathbb{R}^n$ is harmonic of class C^2, the first variation of E gives

$$0 = \langle \varphi, DE(u) \rangle = \int_{T^2} \nabla u \nabla \varphi dx = -\int_{T^2} \Delta u \varphi dx,$$

for all doubly periodic $\varphi \in C^1(\mathbb{R}^2;\mathbb{R}^n)$ satisfying the condition

$$\varphi(x) \in T_{u(x)}N \text{ for all } x \in \mathbb{R}^2.$$

That is, $-\Delta u(x)$ is orthogonal to the tangent space of N at the point $u(x)$, for any $x \in T^2$; in symbols:

$$-\Delta u(x) = \sum_{i=k+1}^{n} \lambda_i(x)\nu_i(u(x)) \perp T_{u(x)}N \quad \text{for all } x \in T^2,$$

where $\nu_{k+1},\dots,\nu_n$ is a smooth local orthonormal frame for the normal bundle $TN^\perp$ near $u(x)$, and where $\lambda_{k+1},\dots,\lambda_n$ are scalar functions. In the general case, the Laplace operator must be replaced by the Laplace-Beltrami operator Δ_Σ on Σ.

For further illustration, consider the case $N = S^{n-1} \subset \mathbb{R}^n$. In this case, if $u : T^2 \to S^{n-1} \subset \mathbb{R}^n$ is of class C^2 and harmonic, it follows that

$$-\Delta u = \lambda u$$

for some continuous function $\lambda : T^2 \to \mathbb{R}$. Multiplying this relation by u and noting that $|u| \equiv 1$, $u \cdot \nabla u \equiv 0$, we see that

$$\lambda = -\text{div}(u \cdot \nabla u) + |\nabla u|^2 = |\nabla u|^2;$$

that is, harmonic maps into spheres satisfy the relation

$$-\Delta u = u|\nabla u|^2.$$

For general target manifolds N we may proceed similarly. Given $x \in \Sigma$, fix $i \in \{k+1,\dots,n\}$. Then, since $\partial_\alpha u = \frac{\partial}{\partial x_\alpha}u \in T_uN$ for any α, we have

$$\begin{aligned}\lambda_i &= -\langle \Delta u, \nu_i \circ u\rangle \\ &= -\text{div}\langle \nabla u, \nu_i \circ u\rangle + \langle \nabla u, (d\nu_i \circ u)\cdot \nabla u\rangle \\ &= \langle \nabla u, (d\nu_i \circ u)\cdot \nabla u\rangle.\end{aligned}$$

That is, we have

$$-\Delta u = \sum_{i=k+1}^{n} \lambda_i \nu_i \circ u = A(u)(\nabla u, \nabla u),$$

where $A(u) : T_uN \times T_uN \to (T_uN)^\perp$ given by

$$A(p)(\xi,\eta) = \sum_{i=k+1}^{n} \nu_i(p)A_i(p)(\xi,\eta)\ ,\ A_i(p)(\xi,\eta) = \langle \xi, d\nu_i(p)\eta\rangle$$

denotes the second fundamental form of N, for $p \in N$, $\xi, \eta \in T_pN$. Here, for clarity, we let $\langle\cdot,\cdot\rangle$ denote the scalar product in $\mathbb{R}^n$. For general domains, similarly the harmonic map equation reads

$$-\Delta_\Sigma u = A(u)(\nabla u, \nabla u)_\Sigma \perp T_uN,$$

with

$$A(u)(\nabla u, \nabla u)_\Sigma = \sum_{\alpha,\beta} \gamma^{\alpha\beta} A(u)(\partial_\alpha u, \partial_\beta u).$$

We note that by compactness of N the coefficients of the form A are uniformly bounded.

3 Bochner identity

Upon differentiating the harmonic map equation and taking the scalar product with the components of ∇u, we obtain an equation for the energy density. In particular, in the case of harmonic maps $u : T^2 \to S^{n-1} \hookrightarrow \mathbb{R}^n$ we obtain

$$\begin{aligned} -\Delta e(u) + |\nabla^2 u|^2 &= -\nabla(\Delta u)\cdot\nabla u \\ &= |\nabla u|^4 + u(\nabla|\nabla u|^2)\cdot\nabla u = |\nabla u|^4 = 4e(u)^2. \end{aligned}$$

Here we also used that $u \cdot \nabla u = \frac{1}{2}\nabla|u|^2 = 0$.

Similarly, for a general target manifold $N \subset \mathbb{R}^n$ we obtain

$$-\Delta e(u) + |\nabla^2 u|^2 = \sum_{i=k+1}^{n} (A_i(u)(\nabla u, \nabla u))^2 \le Ce(u)^2.$$

For a general domain manifold Σ, additional terms related to the curvature of Σ appear. If we do not care about the precise form of these terms we then obtain the inequality

$$-\Delta_\Sigma e(u) + c|\nabla^2 u|^2 \le C|\nabla u|^4 + C|\nabla u|^2$$

with constants $c > 0$, C depending on N and Σ.

The most interesting variant of the Bochner identity results if we work intrinsically on the manifold N and use covariant differentiation instead of taking the ordinary gradient. Then we obtain the differential inequality

$$-\Delta_\Sigma e(u) \leq \kappa_N |\nabla u|^4 + C|\nabla u|^2$$

for the energy density of u, where κ_N denotes an upper bound for the sectional curvature of N, and where C denotes a constant depending only Σ and N.

To fix ideas, in the following we may always think of mappings $u : T^2 \to S^{n-1} \subset \mathbb{R}^n$.

4 The Homotopy Problem and its Functional Analytic Setting

A natural generalization of Dirichlet's problem for harmonic functions now is the following:

Problem 1. Given a map $u_0 : \Sigma \to N$, is there a harmonic map u homotopic to u_0?

As in the case of a scalar function $u : \Omega \to \mathbb{R}$ we may attempt to approach this problem by direct methods; that is, we study the existence of local minimizers of E: Denote

$$H^{1,2}(\Sigma; N) = \{u \in H^{1,2}(\Sigma; \mathbb{R}^n) \ ; \ u(\Sigma) \subset N \text{ a.e.}\},$$

where

$$H^{1,2}(\Sigma; \mathbb{R}^n) = \{u : \Sigma \to \mathbb{R}^n; u, \nabla u \in L^2\},$$

the space of $H^{1,2}$-mappings into N. (If $\Sigma = T^2$, then $H^{1,2}(\Sigma; N)$ is the space of mappings $u \in H^{1,2}_{\text{loc}}(\mathbb{R}^2; N)$ of period 1 in both variables, restricted to a fundamental domain.) Then the functional E is weakly lower semi-continuous and coercive on $H^{1,2}(\Sigma; N)$ with respect to $H^{1,2}(\Sigma; \mathbb{R}^n)$.

Moreover, by a result of Schoen-Uhlenbeck [ScU; Section IV] we have:

Theorem 4.1. *(1°) The space* $C^\infty(\Sigma; N)$ *of smooth maps* $u : \Sigma \to N \subset \mathbb{R}^n$ *is dense in* $H^{1,2}(\Sigma; N)$.

(2°) The homotopy class of an $H^{1,2}(\Sigma; N)$*-map is well defined.*

Motivated by 4.1 one could attempt to solve the homotopy problem, Problem 1, by minimizing E in the homotopy class of u_0. However, while $H^{1,2}(\Sigma; N)$ is weakly closed in the topology of $H^{1,2}(\Sigma; \mathbb{R}^n)$, in general this will not be the case for homotopy classes of non-constant maps. Therefore, the direct method fails to be applicable for solving Problem 1. In fact, the infimum of E in a given homotopy class need not be attained, see [W].

5 Existence and Non-Existence Results

In fact, Problem 1 need not always have an affirmative answer. This result is due to Eells-Wood [EW]:

Theorem 5.1. *Any harmonic map* $u \in C^1(T^2; S^2)$ *necessarily has topological degree* $\neq 1$.

In particular, there is no harmonic map homotopic to a map $u_0 : T^2 \to S^2$ of degree $+1$. This means that we may encounter some lack of compactness in attempting to find critical points of E.

However, compactness can be restored under suitable conditions. Imposing a restriction on the sectional curvature of the target manifold, Eells-Sampson [ES] have obtained the following result:

Theorem 5.2. *Suppose that the sectional curvature* κ_N *of* N *is non-positive. Then for any map* $u_0 : \Sigma \to N$ *there exists a harmonic map homotopic to* u_0.

We sketch the main idea of the proof. In order to overcome the difficulties mentioned above, Eells and Sampson consider the evolution problem

$$u_t - \Delta_\Sigma u = A(u)(\nabla u, \nabla u)_\Sigma \perp T_u N \text{ in } \Sigma \times \mathbb{R}_+ \tag{1}$$

$$u|_{t=0} = u_0 \tag{2}$$

associated with E. In fact, equation (1) defines the L^2-gradient flow for E; in particular, we have the *energy inequality*

$$\int_0^T \int_\Sigma |\partial_t u|^2 d\Sigma dt + E(u(T)) \leq E(u_0),$$

for all $T > 0$.

Analogous to the stationary case, for (1) there holds the Bochner-type inequality

$$(\partial_t - \Delta_\Sigma)e(u) + c|\nabla^2 u|^2 \leq C|\nabla u|^4 + C|\nabla|^2.$$

Moreover, the bound for the leading term on the right can be improved and we obtain the differential inequality

$$(\partial_t - \Delta_\Sigma)e(u) \leq \kappa_N |\nabla u|^4 + C|\nabla u|^2, \tag{3}$$

where κ_N again denotes an upper bound for the sectional curvature of N.

Now, if $\kappa_N \leq 0$, estimate (3) implies a linear differential inequality for the energy density, and we obtain the existence of a global solution $u \in C^2(\Sigma \times \mathbb{R}^+, N)$ to the evolution problem (1), (2). Moreover, by the weak Harnack inequality for sub-solutions of parabolic equations and the energy inequality, the maximum of $|\nabla u|$ may be a priori bounded in terms of the initial energy. Again by the energy inequality, we can find a sequence of numbers $t_m \to \infty$ such that $\partial_t u(t_m) \to 0$ in L^2 as $m \to \infty$ and it follows that $(u(t_m))$ converges to a harmonic map.

Surprisingly, a topological condition on the target manifold may also suffice to solve the homotopy problem. The following result was obtained independently by Lemaire [Le] and Sacks-Uhlenbeck [SU]:

Theorem 5.3. *If $\pi_2(N) = 0$, then for any $u_0 \in H^{1,2}(\Sigma; N)$ there is a smooth harmonic map homotopic to u_0.*

A proof of this result may be given based on the analysis of the "L^2-gradient flow" (1), (2).

6 The Evolution of Harmonic Maps

Now we establish that (1), (2) admits a global weak solution for arbitrary initial data $u \in H^{1,2}(\Sigma; N)$, without any topological or geometric restrictions on the target manifold. However, by Theorem 5.1 we cannot expect in general the existence of a smooth global solution, converging asymptotically to a harmonic map. Let $\exp_x : T_x\Sigma \to \Sigma$ denote the exponential map at a point $x \in \Sigma$. (If $\Sigma = T^2$, then $\exp_x(y) = x + y$.)

Theorem 6.1. *For any $u_0 \in H^{1,2}(\Sigma; N)$ there exists a distribution solution $u : \Sigma \times \mathbb{R}^+ \to N$ of (1) which is smooth on $\Sigma \times \mathbb{R}^+$ away from at most finitely many points $(\bar{x}_k, \bar{t}_k)$, $1 \le k \le K$, $0 \le \bar{t}_k \le \infty$, which satisfies the energy inequality $E(u(t)) \le E(u(s))$ for all $0 \le s \le t$, and which assumes its initial data continuously in $H^{1,2}(\Sigma; N)$. The solution u is unique in this class.*

At a singularity $(\bar{x}, \bar{t})$ a smooth harmonic map $\bar{u} : S^2 \cong \overline{\mathbb{R}^2} \to N$ separates in the sense that for sequences $x_m \to \bar{x}$, $t_m \nearrow \bar{t}$, $R_m \searrow 0$ as $m \to \infty$ the family

$$u_m(x) \equiv u(\exp_{x_m}(R_m x), t_m) \to \tilde{u}, \quad \text{in } H^{2,2}_{\text{loc}}(\mathbb{R}^2; N) \ ,$$

where $\tilde{u}$ has finite energy and extends to a smooth harmonic map $\bar{u} : S^2 \cong \overline{\mathbb{R}^2} \to N$.

As $t_m \to \infty$ suitably, the sequence of maps $u(\cdot, t_m)$ converges weakly in $H^{1,2}(\Sigma; N)$ to a smooth harmonic map $u_\infty : \Sigma \to N$, and smoothly away from finitely many points $(\bar{x}_k, \bar{t}_k = \infty)$. Moreover, we have

$$E(u_\infty) \le E(u_0) - K\varepsilon_0 \ ,$$

where K is the number of singularities and where

$$\varepsilon_0 = \inf\{E(u) \ ; \ u \in C^1(S^2; S^2) \text{ is non-constant and harmonic}\} > 0$$

is a constant depending only on the geometry of N. In particular, the number of singularities of u is a priori bounded, $K \le \varepsilon_0^{-1} E(u_0)$.

Theorem 6.1 implies Theorem 5.3; see [St4]. The proof of Theorem 6.1 is based on a Sobolev type inequality, due to Ladyzhenskaya [L]; see [St2].

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