NOTES ON HODGE THEORY ON KÄHLER MANIFOLDS

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1. Introduction

Differentiable manifolds are familiar objects, defined as topological spaces locally diffeomorphic with $\mathbb{R}^n$. A natural extension of the concept is to define a *complex manifold* to be a topological space locally diffeomorphic with $\mathbb{C}^n$. The transition functions of coordinate charts determine the local properties of the manifold. In the case of a real differentiable manifold, we required the local diffeomorphisms to admit smooth transition functions. In the case of complex manifolds, we ask that the transition functions be holomorphic.

In the course of analyzing the structure of a complex manifold, we notice that the complex coordinate maps define an action of the imaginary numbers on each tangent space. The complex structure of the complex manifold induces endomorphisms on the real tangent spaces of the underlying real differentiable manifold. For each point $x \in M$, we write the induced endormorphism on T_xM as J_x. If J_x varies smoothly along the manifold, we can define J to be a $(1,1)$-tensor field on M which acts as J_x at each point $x \in M$. This tensor field has the crucial property that $J_x^2 = -1$, and is called an *almost complex structure.*

Given a complex manifold, we get an almost complex structure on the underlying real manifold, as above. We ask the converse question: given a real manifold, does it admit an almost complex structure J, and is it the underlying manifold for a complex manifold that induces J. The answer in general is no, but easily checked conditions allow us to determine when this is the case.

Every real manifold admits a Riemannian metric, and a unique connection which is compatible with this metric. Given a manifold with an almost complex structure, we can ask how the extra structure of the J tensor field interacts with a metric and connection. Exploring the dependencies leads us to the definition of a *Kähler metric* and *Kähler manifolds.* A Kähler metric on a complex manifold is roughly analogous to a Riemannian metric on a real manifold: a Kähler holonomy group is a subgroup of the unitary group, the "complex version" of the orthogonal holonomy group of a Riemannian metric. Kähler metrics are pretty restrictive, and impose a rigid local structure on the manifold.

The rigidity of a Kähler metric gives us extra identities in Hodge theory, and gives us two useful decompositions of the comohomology of a Kähler manifold: the Hodge and Lefschetz decompositions. Both of these depend on the class of *harmonic forms*, the kernel of the *Laplace-Beltrami operator.*

These introductory notes were written for my own benefit, and to satisfy a class requirement. The theorems, proofs and explanations are taken liberally from Bryant [1], Griffiths & Harris [2], Joyce [3] and Kobayashi & Nomizu [4]. Some paragraphs and sections are lifted verbatim. Neither the math nor the presentation are new.

Date: December 4, 2003.

2. ALMOST COMPLEX STRUCTURES ON DIFFERENTIABLE MANIFOLDS

2.1. **Coordinates And Cotangent Space On $\mathbb{C}^n$.** Let $(z_1, \ldots, z_n)$ be a coordinate system on $\mathbb{C}^n$. Then we can write $z_i = x_i + \sqrt{-1}\, y_i$ for each i. The cotangent space to a point in $\mathbb{C}^n$ is spanned by $\{dx_i, dy_i\}$, but we'll find it more convenient to work with the following change of basis

$$dz_i = dx_i + \sqrt{-1}\, dy_i, \qquad d\bar{z}_i = dx_i - \sqrt{-1}\, dy_i.$$

The dual basis for the tangent space becomes

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\,\frac{\partial}{\partial y_i}\right), \qquad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\,\frac{\partial}{\partial y_i}\right). \tag{1}$$

In terms of these bases, the formula for the total differential of a complex-valued function is

$$df = \sum_i \frac{\partial}{\partial z_i} dz_i + \sum_i \frac{\partial}{\partial \bar{z}_i} d\bar{z}_i.$$

We write the first term as ∂f and the second as $\bar{\partial} f$. For a function $f : \mathbb{C}^n \to \mathbb{C}^n$, the Cauchy-Riemann equations are equivalent to $\frac{\partial f}{\partial \bar{z}_i} = 0$, as you can check directly from (1). This is equivalent to the statement that $\bar{\partial} f = 0$.

2.2. **Manifolds And Their Tangent Spaces.** An n-dimensional real manifold is a topological space that is locally homeomorphic with $\mathbb{R}^n$. We define complex manifolds by analogy.

Definition 2.1. *An* n-dimensional complex manifold *is a* $2n$*-dimensional differential manifold together with an open cover* U_α *and coordinate maps* $\phi : U_\alpha \to \mathbb{C}^n$ *such that the maps* $\phi_\alpha \circ \phi_\beta^{-1}$ *are holomorphic on* $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$.

Note the important fact that an n-dimensional complex manifold M is simultaneously a $2n$-dimensional real manifold.

Definition 2.2. *An* almost complex structure *is a* $(1,1)$*-tensor field on a differential manifold where* J *is an endomorphism of* T_pM *for each* $p \in M$ *and* $J^2 = -1$. *An* almost complex manifold *is a differential manifold together with an almost complex structure.*

A short argument shows that an almost complex manifold is orientable and must have even dimension.

Every complex manifold M has a natural almost complex structure. Suppose $p \in M$ and $(z_1, \ldots, z_n)$ is a local complex coordinate system for p, where $z_i = x_i + \sqrt{-1}\, y_i$. Then $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\}$ is a basis for T_pM, and we define

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \qquad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}.$$

We're left with an obvious question: when is a manifold M with an almost complex structure J actually a complex manifold? The answer lies in a $(1,2)$-tensor on M called the *torsion* of J, also called the *Nijnhuis tensor*. We define the torsion by

$$N(X,Y) = 2\Big\{[JX, JY] - [X,Y] - J[X, JY] - J[JX, Y]\Big\}.$$

A theorem by Newlander and Nirenberg shows that J actually defines a complex structure on M iff the torsion is 0.

We turn now to the tangent spaces of an almost complex manifold. For real manifolds we have one commonly encountered tangent space, but for complex manifolds we have three.

(i) Since an almost complex manifold M is an even-dimensional real manifold, we can choose coordinates $(x_1, y_1, \dots, x_n, y_n)$ around a point $p \in M$. Then $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\}$ is a basis for the usual tangent space T_pM, and we can write

$$T_pM = \mathbb{R}\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right\}.$$

This is the *real tangent space*, so-called because it is a real vector space, and when we want to distinguish it from the other tangent spaces we will write it as $T_p^{\mathbb{R}}M$.

(ii) Working with complex manifolds, it will sometimes be convenient to have tangent spaces that are complex vector spaces. To obtain them, we simply complexify the real tangent spaces, so $T_p^{\mathbb{C}}M = T_p^{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$. With a change of basis for $T_p^{\mathbb{R}}M$ from $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\}$ to $\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\}$, we can write

$$\begin{aligned} T_p^{\mathbb{C}}M &= \mathbb{C}\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right\} \\ &= \mathbb{C}\left\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right\}. \end{aligned}$$

Notice that if the underlying real manifold of M is $2n$-dimensional, then the underlying real vector space of $T_p^{\mathbb{C}}M$ is $4n$-dimensional.

(iii) Since $J^2 = -1$, J is nondegenerate and its eigenvalues are $\{-\sqrt{-1}\,, \sqrt{-1}\,\}$. Thus we can split $T_p^{\mathbb{C}}M = T_p^{1,0}M \oplus T_p^{0,1}M$, where $T_p^{1,0}M$ is the $\sqrt{-1}$-eigenspace and $T_p^{0,1}M$ is the $-\sqrt{-1}$-eigenspace. A proposition tells us that $T_p^{1,0}M = \left\{X - \sqrt{-1}\,JX\right\}$ and $T_p^{0,1}M = \left\{X + \sqrt{-1}\,JX\right\}$ for X a real tangent vector, *i.e.* $X \in T_p^{\mathbb{R}}M$. Thus we can also write our new tangent spaces as

$$T_p^{1,0}M = \mathbb{C}\left\{\frac{\partial}{\partial z_i}\right\}, \qquad T_p^{0,1}M = \mathbb{C}\left\{\frac{\partial}{\partial \bar{z}_i}\right\}.$$

$T_p^{1,0}M$ is called the *holomorphic tangent space* and $T_p^{0,1}M$ is called the *antiholomorphic tangent space.*

The spaces $T_p^{1,0}M$ and $T_p^{0,1}M$ are isomorphic, and the isomorphism between them is called *conjugation.* Thus $T_p^{0,1}M$ is sometimes written $\overline{T_p^{1,0}M}$. We therefore know that both the holomorphic and antiholomorphic tangent spaces are $2n$-dimensional as real vector spaces, and thus we have an $\mathbb{R}$-linear isomorphism between $T_p^{\mathbb{R}}M$ and $T_p^{1,0}M$.

We call a map $f : M \to N$ between almost complex manifolds a *holomorphic* map if it maps the holomorphic tangent space at $p \in M$ into the holomorphic tangent space at $f(p) \in N$, i.e. $f_*(T_p^{1,0}M) \subset T_{f(p)}^{1,0}N$ for all $p \in M$. If f is just a map from $\mathbb{C}$ to $\mathbb{C}$, then the holomorphic tangent space at any point z_0 is $\mathbb{C}\left\{\frac{\partial}{\partial z}|_{z_0}\right\}$ and

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z},$$

so mapping the source holomorphic tangent space into the target holomorphic tangent space is equivalent to $\frac{\partial f}{\partial \bar{z}} = 0$, which is the usual condition that f be holomorphic.

2.3. **Differential Forms And Cohomology.**

We want to understand differential forms on complex manifolds. We'll be looking at complex-valued forms on complex manifolds, and they will take vectors from the complexified tangent space $T_z^{\mathbb{C}}M$. Since we can split the tangent space as $T_z^{\mathbb{C}}M = T_z^{1,0}M \oplus T_z^{0,1}$, our complex differential forms will reflect this structure. First, de Rham cohomology.

Definition 2.3. *If M is a differential manifold, let $A^n(M)$ denote the complex-valued n-forms on M, and let $Z^n(M)$ denote the complex-valued closed n-forms on M. Then the* nth de Rham cohomology group *is*

$$H^n_{dR}(M) = \frac{Z^n(M)}{dA^{n-1}(M)}.$$

If $H^n_{dR}(M, \mathbb{R})$ denotes the usual real-valued de Rham cohomology, then $H^n_{dR}(M) = H^n_{dR}(M, \mathbb{R}) \otimes \mathbb{C}$.

Now we assume that M is a complex manifold. The holomorphic and antiholomorphic tangent spaces give us a decomposition of the complex n-forms. We write $T_z^{\mathbb{C}^*}(M)$ for the complex cotangent bundle. With some linear algebra,

$$\begin{aligned} \bigwedge^n T_z^{\mathbb{C}^*}(M) &= \bigwedge^n \left(T_z^{1,0^*}(M) \oplus T_z^{0,1^*}(M)\right) \\ &= \bigoplus_{p+q=n} \left\{\bigwedge^p T_z^{1,0^*}(M) \otimes \bigwedge^q T_z^{0,1^*}(M)\right\}. \end{aligned}$$

This gives us a decomposition at each individual tangent space. We define

$$A^{p,q}(M) = \left\{\phi \in A^n(M) : \phi(z) \in \bigwedge^p T_z^{1,0^*}(M) \otimes \bigwedge^q T_z^{0,1^*}(M) \text{ for all } z \in M\right\}$$

so

$$A^n(M) = \bigoplus_{p+q=n} A^{p,q}(M).$$

If $\phi \in A^{p,q}(M)$, we say ϕ is *of type* (p,q), and then

$$\begin{aligned} d\phi &\in \left(\bigwedge^p T_z^{1,0^*}(M) \otimes \bigwedge^q T_z^{0,1^*}(M)\right) \wedge T_z^{\mathbb{C}^*}(M) \\ &\in A^{p+1,q}(M) \oplus A^{p,q+1}(M). \end{aligned}$$

We define operators

$$\begin{aligned} \partial &: A^{p,q}(M) \to A^{p+1,q}(M) \\ \bar{\partial} &: A^{p,q}(M) \to A^{p,q+1}(M) \end{aligned}$$

by projection, so

$$d = \partial + \bar{\partial}.$$

If $\phi = \sum_{\{I=p, J=q\}} \phi_{IJ} dz^I \wedge d\bar{z}^J$, where I and J are multi-indices, then writing out $\partial\phi$ and $\bar{\partial}\phi$ explicitly,

$$\begin{aligned} \partial\phi &= \sum_{I,J,i} \frac{\partial \phi_{IJ}}{\partial z} dz_i \wedge dz_I \wedge d\bar{z}_J \\ \bar{\partial}\phi &= \sum_{I,J,j} \frac{\partial \phi_{IJ}}{\partial \bar{z}} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J. \end{aligned}$$

Note that since partial derivatives commute, $\bar{\partial}^2 = 0$. Thus we have a $\bar{\partial}$-cohomology theory. Let $Z^{p,q}_{\bar{\partial}}(M)$ be the $\bar{\partial}$-closed (p,q)-forms.

Definition 2.4. *The* (p,q) Dolbeault cohomology group *is the quotient*

$$H^{p,q}_{\bar{\partial}}(M) = \frac{Z^{p,q}_{\bar{\partial}}(M)}{\bar{\partial}A^{p,q-1}(M)}.$$

3. Metrics And Kähler Manifolds

3.1. **Hermitian Metrics.** Let M be a complex manifold of dimension n.

Definition 3.1. *A* hermitian metric *on M is a positive definite hermitian inner product at each point of M, which also varies smoothly over M.*

An inner product h on a complex vector space V is *hermitian* if it is linear in the first term, and conjugate linear in the second, so $h(\alpha x, \beta y) = \alpha\bar{\beta}\, h(x,y)$. Another way of looking at this is to say that $h : V \oplus \overline{V} \to \mathbb{C}$ is linear on each factor of $V \oplus \overline{V}$, where $\overline{V}$ is the conjugate vector space.

If $(z_1, \ldots, z_n)$ is a local coordinate system around $z \in M$, and h is a hermitian metric on M, we can write it as

$$h = \sum h_{ij} dz_i \otimes d\bar{z}_j$$

where $h_{ij}(z) = (\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})_z$ is a smooth function, $(\quad,\quad)_z$ is a hermitian inner product, and $h_{ij} = \overline{h_{ji}}$. By the Gram-Schmidt process, at each $z \in M$ we can find an orthonormal basis $\{\varphi_1(z) \ldots \varphi_n(z)\}$ on $T^{1,0}_z(M)$ for the inner product $(\ ,\ )_z$. With this orthonormal basis we can rewrite the metric as

$$\begin{aligned} ds^2 &= \sum h_{ij}(z) dz_i \otimes d\bar{z}_j \\ &= \sum \varphi_i \otimes \overline{\varphi}_j. \end{aligned}$$

A hermitian metric is complex-valued. Looking at its real and imaginary parts splits the metric into a symmetric and antisymmetric part. Restricting our metric to the holomorphic tangent space (which is isomorphic to the underlying real tangent space), we get a Riemannian metric and a special differential form on the underlying manifold. Explicity, if $\{\varphi_1, \ldots, \varphi_n\}$ is a unitary coframe for ds^2 and if we write $\varphi_i = \alpha_i + \sqrt{-1}\,\beta_i$ where α_i, β_i are real differential forms, then

$$\begin{aligned} ds^2 &= \sum (\alpha_i + \sqrt{-1}\,\beta_i) \otimes (\alpha_i - \sqrt{-1}\,\beta_i) \\ &= \sum (\alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i) + \sqrt{-1}\, \sum (-\alpha_i \otimes \beta_i + \beta_i \otimes \alpha_i). \end{aligned}$$

The first term is a symmetric, nondegenerate bilinear form, so $\mathrm{Re}(ds^2)$ is a Riemannian metric on the holomorphic tangent space. The second term is an alternating bilinear form, i.e. a real differential 2-form. We can consider these as bilinear forms on the underlying real manifold, using the isomorphism between $T^{\mathbb{R}}_z(M)$ and $T^{1,0}_z(M)$. We call $\omega = -\frac{1}{2}\mathrm{Im}(ds2)$ the *associated* $(1,1)$*-form* of the metric. In terms of the above,

$$\begin{aligned}\omega &= -\frac{1}{2}\sum(-\alpha_i \otimes \beta_i + \beta_i \otimes \alpha_i)\\ &= \sum \alpha_i \wedge \beta_i\\ &= \frac{\sqrt{-1}}{2}\sum \varphi_i \wedge \overline{\varphi}_i.\end{aligned}$$

Conversely, given an associated $(1,1)$-form ω we can reconstruct the original hermitian metric.

Example 3.2. *On $\mathbb{C}^n$, the standard hermitian metric is the usual one given by $ds^2 = \sum dz_i \otimes d\bar{z}_i$.*

We call a hermitian metric a *Kähler metric* if $d\omega = 0$, in which case ω is called a *Kähler form.* A complex manifold with a Kähler metric is called (unsurprisingly) a *Kähler manifold.* Note that since ω must be nondegenerate, a Kähler manifold must also be symplectic. The Kähler condition may seem somewhat random, and it certainly doesn't give much geometrical insight, but it is very powerful and very common. To give some better motivation and geometrical insight on it, I'll digress from the main development here and give some alternative formulations. We'll start with the holonomy of a Riemannian manifold.

3.2. Kähler Metrics From The Holonomy Classification.

The fundamental theorem of Riemannian geometry is that on any Riemannian manifold M we have a canonical connection: the Levi-Civita connection. This connection defines the parallel transport of a vector v along a piecewise-smooth curve $\gamma : [0,1] \to M$. If we fix $x \in M$ and let γ be a loop at x so $\gamma(0) = \gamma(1) = x$, then the parallel transport of $v \in T_x(M)$ defines an endomorphism on T_xM, which we denote P_γ. The composition of two endomorphisms $P_\gamma \circ P_\beta = P_{\gamma\beta}$ is defined by reparametrizing $\gamma\beta$ so that its domain is $[0,1]$. If we define $\gamma^{-1} : [0,1] \to M$ by $\gamma^{-1}(t) = \gamma(1-t)$, then one can check that $P_{\gamma^{-1}}$ is an inverse for P_γ. Thus the collection of endomorphisms P_γ is a group. Since P_γ is invertible, $P_\gamma \in \mathrm{GL}(T_xM)$.

Definition 3.3. *Let (M,g) be a Riemannian manifold, and fix $x \in M$. Define a* loop at x *to be a piecewise-smooth curve $\gamma : [0,1] \to M$ such that $\gamma(0) = \gamma(1) = x$. The Levi-Civita connection defines an invertible endomorphism P_γ on T_xM by parallel transport. We define the* holonomy group of g at x *to be*

$$\mathrm{Hol}_x(g) = \{P_\gamma : \gamma \text{ is a loop based at } x\} \subset \mathrm{GL}(T_xM).$$

If M is connected, then there exists a path τ between any two points $x, y \in M$. Taking P_τ to be the isomorphism between T_xM and T_yM given by parallel translation, it's clear that if γ is a loop at y, then $\tau\gamma\tau^{-1}$ is a loop at x. Thus if $P_\gamma \in \mathrm{Hol}_y(g)$, then $P_{\tau\gamma\tau^{-1}} = P_\tau \circ P_\gamma \circ P_{\tau^{-1}} \in \mathrm{Hol}_x(M)$, and in particular $P_\tau\,\mathrm{Hol}_y(g)\,P_{\tau^{-1}} = \mathrm{Hol}_x(g)$. Thus, up to conjugacy in $\mathrm{GL}(n,\mathbb{R})$ (n is the complex dimension), the holonomy of a Riemannian manifold is independent of basepoint. Therefore we can define $\mathrm{Hol}(g)$ to be the conjugacy class of $\mathrm{Hol}_x(g)$ in $\mathrm{GL}(n,\mathbb{R})$ for any $x \in M$.

There is a subgroup $\mathrm{Hol}^0(g)$ of the general holonomy group $\mathrm{Hol}(g)$ called the *restricted holonomy group* which is the holonomy group generated by null-homotopic loops. There is a surjective group homomorphism $\phi : \pi_1(M) \to \mathrm{Hol}(g)/\mathrm{Hol}^0(g)$, so if M is simply-connected, $\mathrm{Hol}^0(g) = \mathrm{Hol}(g)$. From here on I will concentrate on simply-connected manifolds when talking about holonomy groups, in order to avoid global topological issues and to concentrate on local properties of the metric.

It is a fact (which I won't prove here) that if (M, g) is a simply-connected Riemannian manifold which can be written as a product $(M_1, g_1) \times (M_2, g_2)$, then the holonomy group $\mathrm{Hol}(g) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2)$. So in studying Riemannian holonomy groups, we can concentrate on those manifolds which are *irreducible*—those that can't be decomposed as a product.

Now in order to classify Riemannian manifolds by their holonomy groups, we just need to introduce the notion of symmetric spaces.

Definition 3.4. *A Riemannian manifold (M, g) is a* Riemannian symmetric space *if for every $p \in M$, there exists an (involutive) isometry $s_p : M \to M$ such that s_p^2 is the identity and such that p is an isolated fixed point of s_p. We say (M, g) is* locally symmetric *if for every $p \in M$ there exists an open neighborhood U_p and an involutive isometry $s_p : U_p \to U_p$ with fixed point p. (M, g) is* nonsymmetric *if it is not locally symmetric.*

Élie Cartan introduced and classified Riemannian symmetric spaces (and their holonomies) in 1925. Thus we're left to classify the nonsymmmetric Riemannian manifolds. This was first accomplished by Berger in 1955, who provided a list of the possible holonomies, which was later refined by Alekseevskii and also by Brown and Gray.

Theorem 3.5 (Berger)**.** *Suppose M is a simply-connected manifold of dimension n, and g is a Riemannian metric on M that is irreducible and nonsymmetric. Then exactly one of the following seven cases holds.*

(i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$
(ii) $n = 2m, \mathrm{Hol}(g) = \mathrm{U}(m)$ *(Kähler)*
(iii) $n = 2m, \mathrm{Hol}(g) = \mathrm{SU}(m)$ *(Calabi-Yau)*
(iv) $n = 4m, \mathrm{Hol}(g) = \mathrm{Sp}(m)$ *(Hyperkähler)*
(v) $n = 4m, \mathrm{Hol}(g) = \mathrm{Sp}(m)\mathrm{Sp}(1)$ *(Quaternionic Kähler)*
(vi) $n = 7, \mathrm{Hol}(g) = G_2$
(vii) $n = 8, \mathrm{Hol}(g) = \mathrm{Spin}(7)$

Metrics g with $\mathrm{Hol}(g) \subset \mathrm{SU}(m)$ are called *Kähler metrics.* This definition is equivalent to our earlier definition, although showing this requires some work. This formulation makes the Kähler condition a little less random. Kähler manifolds are simply a particular class of Riemannian manifolds given by the holonomy classification. Note that Calabi-Yau and hyperkähler metrics are also Kähler, since $\mathrm{Sp}(m) \subset \mathrm{SU}(2m) \subset \mathrm{U}(2m)$.

3.3. **Other Forms Of The Kähler Condition.**

The holonomy classification gives us a natural way of seeing where Kähler manifolds come from, and the definition of a Kähler manifold as a hermitian manifold with a closed associated $(1, 1)$-form gives us a relatively easy way of checking if a metric is Kähler. But there are additional equivalent conditions that are sometimes useful.

Another useful way of looking at Kähler metrics is to view them as metrics which locally look like the Euclidean metric, up to order 2.

Definition 3.6. *A metric ds^2 on M* osculates[1] to order k *to the Euclidean metric on $\mathbb{C}^n$ if for every $z_0 \in M$ we can find a holomorphic coordinate system $(z_1, \ldots, z_n)$*

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[1] *Osculate* comes from Latin for "to kiss"!

around z_0 *such that*

$$ds^2 = \sum(\delta_{ij} + g_{ij})dz_i \otimes d\bar{z}_j,$$

where g_{ij} *vanishes up to order* k*.*

In this terminology, a Kähler metric is a metric which osculates to the Euclidean metric to order 2. The proof isn't too difficult, but I won't go into it here; it's covered in Griffiths & Harris [2]. This is an extremely useful condition. Anything that can be proven for $\mathbb{C}^n$ with the standard hermitian metric and doesn't depend on derivatives of order higher than 1 will also hold for any Kähler manifold.

There are two convenient equivalent descriptions of a Kähler metric. If h is a hermitian metric and ∇ is the associated Levi-Civita connection, then

- (i) $\nabla J = 0$
- (ii) $\nabla \omega = 0$.

In other words, J and ω are parallel with respect to the Levi-Civita connection.

4. The Hodge Decomposition And The Hard Lefschetz Theorem

The Hodge Theorem for a compact complex manifold is a cornerstone of modern differential geometry. It allows us to identify a unique representative for each cohomology class, with the special property that this unique representative is in the kernel of the Laplacian (which we will define). The extra structure of a Kähler manifold gives us extra symmetries in the Hodge structure of the manifold, and leads to the powerful Lefschetz decomposition. But before we tackle the Hodge theory, I need to introduce the Hodge star operator and the Laplacian.

4.1. **The Laplacian.**

Suppose we have an n-dimensional compact complex manifold M with a hermitian metric h. The metric h induces a metric on every vector bundle $\pi : E \to M$; in particular we have a metric on each fiber of $A^{p,q}(M)$, which we'll denote $(,)_z$ at the fiber E_z. The metric varies smoothly between fibers, so we can extend it to a global metric on $A^{p,q}(M)$ by integration:

$$(\alpha, \beta) = \int_M (\alpha(z), \beta(z))_z \Phi(z),$$

where $\Phi = \frac{\omega^n}{n!}$ is the volume form (remember ω is the associated $(1,1)$-form for h). Notice that this integral is always defined on a compact manifold. If we venture away from compact manifolds, we need to put restrictions on the forms, such as considering only forms with compact support, or more generally forms for which the integral is finite (these are called L^2-forms). We define the *Hodge star operator* to be the map $* : A^{p,q}(M) \to A^{n-p,n-q}(M)$ such that

$$\alpha(z) \wedge *\beta(z) = (\alpha(z), \beta(z))_z \Phi(z)$$

for all $\alpha \in A^{p,q}(M)$. This gives us a compact way of writing the metric as

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta.$$

With the metric, $A^{p,q}(M)$ is a Hilbert space (check or justify), and we can define a formal adjoint for $\bar{\partial}$ which we denote $\bar{\partial}^* : A^{p,q+1}(M) \to A^{p,q}(M)$, so that if $\alpha \in A^{p,q}(M)$ and $\beta \in A^{p,q+1}(M)$, then

$$(\bar{\partial}\alpha, \beta) = (\alpha, \bar{\partial}^*\beta).$$

A form is *coclosed* if $\bar{\partial}^* = 0$.

We can now define the central operator in Hodge theory, the Laplacian (also called the *Laplace-Beltrami* operator).

Definition 4.1. *The* Laplacian *is an operator* $\Delta : A^*(M) \to A^*(M)$ *defined by*

$$\Delta = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

A form α *is* harmonic *if* $\Delta\alpha = 0$*, and we denote the space of harmonic* (p, q)*-forms by* $\mathcal{H}^{p,q}(M) = \ker\, \Delta : A^{p,q}(M) \to A^{p,q}(M)$.

Later, to avoid confusion, we may write this Laplacian as $\Delta_{\bar{\partial}}$, and refer to the $\bar{\partial}$-Laplacian and $\bar{\partial}$-harmonic forms. You can check that a form is harmonic iff it is $\bar{\partial}$-closed and $\bar{\partial}$-coclosed. These forms are also the forms of minimal norm in the Dolbeault cohomology.

We are now ready to state the Hodge Theorem for an n-dimensional compact complex manifold M.

Theorem 4.2 (Hodge Theorem)**.** *Suppose* M *is an* n*-dimensional compact complex manifold. Then*

(i) $\dim\, \mathcal{H}^{p,q}(M) < \infty$

(ii) *the orthogonal projection* $\mathcal{H} : A^{p,q}(M) \to \mathcal{H}^{p,q}(M)$ *is well-defined, and there exists a unique operator* $G : A^{p,q}(M) \to A^{p,q}(M)$ *called the* Green's operator *such that* $G(\mathcal{H}^{p,q}(M)) = 0$*,* G *commutes with* $\bar{\partial}$ *and* $\bar{\partial}^*$*, and*

$$I = \mathcal{H} + \Delta G.$$

One of the major implications of this theorem is that every Dolbeault cohomology class is represented by a unique harmonic form (up to scaling), which we show on page 12. Every harmonic form is automatically closed, so we have an isomorphism between the space of (p, q) Dolbeault cohomology classes and the space of (p, q) harmonic forms.

4.2. **The Hodge Decomposition On A Kähler Manifold.**

The extra structure of a Kähler manifold tells us more about the harmonic forms on it. For this section assume that M is an n-dimensional Kähler manifold with metric ds^2 and associated Kähler form ω. We've defined a bunch of operators on M already. Now we'll define some more. Let d^* be the adjoint of d, let ∂^* be the adjoint of ∂, and let $\Delta_d = dd^* + d^*d$ and $\Delta_\partial = \partial\partial^* + \partial^*\partial$ be the respective Laplacians. As a convenience we define

$$d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$$

and write its associated adjoint as d^{c*}. Note that $d^*d^{c*} = -d^{c*}d^*$. We also have the projection operators

$$\Pi^{p,q} : A^*(M) \to A^{p,q}(M)$$

$$\Pi^r = \bigoplus_{p+q=r} \Pi^{p,q} : A^*(M) \to A^r(M)$$

by type and degree.

The Hodge decomposition of a Kähler manifold follows from a basic identity involving the operator $L : A^{p,q}(M) \to A^{p+1,q+1}(M)$, which we define by

$$L(\eta) = \eta \wedge \omega.$$

Let

$$\Lambda = L^* : A^{p,q}(M) \to A^{p-1,q-1}(M)$$

be its adjoint. The basic identity we want to prove is

$$[\Lambda, d] = -4\pi d^{c*}$$

or equivalently,

$$[L, d^*] = 4\pi d^c.$$

I'll call this the *Kähler identity.* By decomposition of type, the identity is equivalent to

$$[\Lambda, \bar\partial] = -\sqrt{-1}\,\partial^* \quad \text{and} \quad [\Lambda, \partial] = \sqrt{-1}\,\bar\partial^*.$$

Since Λ, d and d^c are real operators, either of these implies the other. Proving $[\Lambda, \partial] = \sqrt{-1}\,\bar\partial^*$ is not intrinsically difficult, but it is messy. We'll leave the dirty work to Griffiths & Harris [2]. They prove the identity on $\mathbb{C}^n$, and then argue that since their proof didn't use any derivatives of the metric of order greater than one, the proof carries over to Kähler manifolds. This identity has major consequences, including the Hodge and Lefschetz decompositions. We lead off with two preliminary consequences. The proofs aren't important, but they're short, so I'll include them.

Proposition 4.3. $[L, \Delta_d] = 0$

Proof. Note that since Δ_d is self-adjoint, this is equivalent to $[\Lambda, \Delta_d] = 0$. By our assumption that M is Kähler,

$$\begin{aligned} d(\eta \wedge \omega) &= d\eta \wedge \omega \pm \eta \wedge d\omega \\ &= d\eta \wedge \omega. \end{aligned}$$

Now we can compute $[L, d]$:

$$\begin{aligned} [L, d]\eta &= Ld\eta - dL\eta \\ &= d\eta \wedge \omega - d(\eta \wedge \omega) \\ &= 0. \end{aligned}$$

Taking the adjoint of both sides, $[\Lambda, d^*] = 0$. Using the Kähler identity,

$$\begin{aligned} \Lambda(dd^* + d^*d) &= (d\Lambda - 4\pi d^{c*})d^* + d^*\Lambda d \\ &= d\Lambda d^* - 4\pi d^{c*}d^* + d^*\Lambda d \\ &= d\Lambda d^* + (4\pi d^* d^{c*} + d^*\Lambda d) \\ &= d\Lambda d^* + d^*(4\pi d^{c*} + \Lambda d) \\ &= d\Lambda d^* + d^*(d\Lambda) \\ &= (dd^* + d^*d)\Lambda. \end{aligned}$$

□

Proposition 4.4. $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$

Proof. From part of the Kähler identity, $\Lambda\partial - \partial\Lambda = \sqrt{-1}\,\bar\partial^*$. We calculate

$$\begin{aligned} \sqrt{-1}\,(\partial\bar\partial^* - \bar\partial^*\partial) &= \partial(\Lambda\partial - \partial\Lambda) - (\Lambda\partial - \partial\Lambda)\partial \\ &= \partial\Lambda\partial - \partial\Lambda\partial \\ &= 0. \end{aligned}$$

So $\partial\bar{\partial}^* = -\bar{\partial}^*\partial$. From part of the Kähler identity,

$$\begin{aligned}\Delta_d &= (\partial+\bar{\partial})(\partial^*+\bar{\partial}^*) + (\partial^*+\bar{\partial}^*)(\partial+\bar{\partial}) \\ &= (\partial\partial^*+\partial^*\partial) + (\bar{\partial}\bar{\partial}^*+\bar{\partial}^*\bar{\partial}) + (\bar{\partial}\partial^*+\partial\bar{\partial}^*+\partial^*\bar{\partial}+\bar{\partial}^*\partial) \\ &= (\partial\partial^*+\partial^*\partial) + (\bar{\partial}\bar{\partial}^*+\bar{\partial}^*\bar{\partial}) \\ &= \Delta_\partial + \Delta_{\bar{\partial}}.\end{aligned}$$

We'll be done if we can show that $\Delta_\partial = \Delta_{\bar{\partial}}$. Examine

$$\begin{aligned}\sqrt{-1}\,\Delta_{\bar{\partial}} &= \bar{\partial}(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\bar{\partial} \\ &= -\left(\partial\Lambda\bar{\partial} - \partial\bar{\partial}\Lambda + \Lambda\bar{\partial}\partial - \bar{\partial}\Lambda\partial\right) \\ &= -\left(\partial(\Lambda\bar{\partial} - \bar{\partial}\Lambda) + (\Lambda\bar{\partial} - \bar{\partial}\Lambda)\partial\right) \\ &= \sqrt{-1}\,(\partial\partial^* + \partial^*\partial) \\ &= \sqrt{-1}\,\Delta_\partial\end{aligned}$$

since $[\Lambda, \bar{\partial}] = -\sqrt{-1}\,\partial^*$. □

Note how both of these propositions rely on the Kähler identities. The second proposition tells us that—up to a factor of 2—on a Kähler manifold the operators $\Delta_d, \Delta_\partial, \Delta_{\bar{\partial}}$ all coincide, and the corresponding harmonic forms are all the same.

Since Δ_∂ and $\Delta_{\bar{\partial}}$ preserve type, so does Δ_d, i.e.

$$[\Delta_d, \Pi^{p,q}] = 0.$$

This commutativity is crucial to the Hodge and Lefschetz decompositions.

Define

$$\begin{aligned}H^{p,q}(M) &= \frac{Z_d^{p,q}(M)}{dA^*(M)\cap Z_d^{p,q}(M)} \\ \mathcal{H}^{p,q}(M) &= \{\varphi \in A^{p,q}(M) : \Delta_d\varphi = 0\} \\ \mathcal{H}^r(M) &= \{\varphi \in A^r(M) : \Delta_d\varphi = 0\}.\end{aligned}$$

Since $A^r(M) = \bigoplus_{p+q=r} A^{p,q}(M)$ and Δ_d commutes with $\Pi^{p,q}$, if $\alpha \in A^r(M)$,

$$\begin{aligned}\Delta_d\alpha &= \Delta_d\left(\alpha^{r,0} + \alpha^{r-1,1} + \cdots + \alpha^{0,r}\right) \\ &= \Delta_d\alpha^{r,0} + \Delta_d\alpha^{r-1,1} + \cdots + \Delta_d\alpha^{0,r}\end{aligned}$$

where $\alpha^{p,q} = \Pi^{p,q}\alpha$. Thus $\Delta_d\alpha = 0$ iff $\Delta_d\alpha^{p,q} = 0$ for each (p,q), and

$$\mathcal{H}^r(M) = \bigoplus_{p+q=r} \mathcal{H}^{p,q}(M). \tag{2}$$

The complex differential forms are the complexification of the real forms, which we can write as $A^*(M) = A^*(M;\mathbb{R})\otimes\mathbb{C}$. Since the differential d is defined on the real forms, Δ_d can be considered as an operator on the real forms which is extended to the complex forms by complex linearity. In the tensor product described above, conjugation just acts on the factor of $\mathbb{C}$. From this description it is clear that Δ_d commutes with conjugation. Since conjugation is an isomorphism between $A^{p,q}(M)$ and $A^{q,p}(M)$, we obtain

$$\mathcal{H}^{p,q}(M) = \overline{\mathcal{H}^{q,p}(M)}. \tag{3}$$

From the Hodge Theorem, we know that if $\varphi \in A^{p,q}(M)$ and $d\varphi = 0$, then

$$\begin{aligned}\varphi &= \mathcal{H}(\varphi) + \Delta_d G(\varphi)\\ &= \mathcal{H}(\varphi) + dd^*G(\varphi) + d^*dG(\varphi)\\ &= \mathcal{H}(\varphi) + dd^*G(\varphi)\end{aligned}$$

since d and G commute. $\mathcal{H}(\varphi)$ is also of type (p,q), so

$$H^{p,q}(M) \cong \mathcal{H}^{p,q}(M). \tag{4}$$

The Hodge Theorem for Δ_d also tells us that $H^*_{dR}(M) \cong \mathcal{H}^*(M)$. Putting this together with (2), (3) and (4), we get

Theorem 4.5 (Hodge Decomposition)**.** *On a compact Kähler manifold M we have*

$$H^r(M) = \bigoplus_{p+q=r} H^{p,q}(M)$$

$$H^{p,q}(M) = \overline{H^{q,p}(M)}.$$

Since $\Delta_d = 2\Delta_{\bar\partial}$, we also have $\mathcal{H}^{p,q}(M) \cong \mathcal{H}^{p,q}_{\bar\partial}(M)$, which means

$$H^{p,q}(M) \cong H^{p,q}_{\bar\partial}(M).$$

An immediate consequence of the Hodge theorem is a strong condition on the Betti numbers of a Kähler manifold.

Proposition 4.6. *On a compact Kähler manifold M the odd Betti numbers must be even.*

This gives us a topological obstruction to a complex manifold admitting a Kähler structure. As an example, start with the usual complex structure on $\mathbb{C}^2 - \{0\}$, and take the quotient by the group action $z \mapsto 2z$. The quotient inherits the complex structure, and is a complex manifold. Topologically this space is equivalent to $S^3 \times S^1$, and is called the *Hopf surface*. By the Künneth formula, $H^*(S^3 \times S^1) = H^*(S^3) \otimes H^*(S^1)$, so $H^1(S^3 \times S^1) \cong \mathbb{C}$, and therefore the manifold can't admit a Kähler structure.

4.3. **The Lefschetz Decomposition On A Kähler Manifold.**

The operators L, Λ and their commutativity with Δ_d on a Kähler manifold give us another decomposition of the cohomology of the manifold. These operators generate a vector space which we can identify with $\mathfrak{sl}_2$. With this identification we get the fantastic fact that the cohomology is an $\mathfrak{sl}_2$-representation!

Here's a blitzprimer[2] on $\mathfrak{sl}_2$, the Lie algebra for SL_2. $\mathfrak{sl}_2$ can be written as the vector space of traceless 2×2 complex matrices with commutator

$$[A,B] = AB - BA.$$

The standard basis for $\mathfrak{sl}_2$ is given by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with relations

$$[H,X] = 2X \qquad [H,Y] = -2Y \qquad [X,Y] = H.$$

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[2] *blitzprimer:* Lightning-fast primer

A *representation* of $\mathfrak{sl}_2$ is a complex vector space V and a Lie algebra homomorphism $\rho : \mathfrak{sl}_2 \to \operatorname{End}(V)$. In this case V is called an $\mathfrak{sl}_2$*-module.* Unless there is some possible confusion, I'll omit the ρ's for the action of $\mathfrak{sl}_2$ on V. A *submodule* is a subspace of V fixed by the action of $\mathfrak{sl}_2$. V is *irreducible* if V has no nontrivial submodules. Irreducible representations of $\mathfrak{sl}_2$ have a particular structure given by the eigenspaces of H, which are called *weightspaces.* If v is in a weightspace for H with eigenvalue λ, then Xv and Yv are also in weightspaces with eigenvalues $\lambda + 2$ and $\lambda - 2$, respectively. We'll do the calculation for Xv (which is very similar to the case of Yv):

$$\begin{aligned} H(Xv) &= [H, X]v + X(Hv) \\ &= (2X)v + X(\lambda v) \\ &= (\lambda + 2)Xv. \end{aligned}$$

Since V is finite-dimensional, H only has a finite number of eigenvalues. This implies that X and Y are nilpotent. We call X *primitive* if v is an eigenvector for H and $Xv = 0$.

I'll now list some important propositions without proof.

Proposition 4.7. *If v is primitive, then V is generated by*

$$v, Yv, Y^2v, \ldots .$$

Since the nonzero elements of V of the form $Y^n v$ all have different eigenvalues, they are linearly independent. If we write the λ-eigenspace of H as V_λ we have

$$H(V_\lambda) = V_\lambda \qquad X(V_\lambda) = V_{\lambda+2} \qquad Y(V_\lambda) = V_{\lambda-2},$$

and

Proposition 4.8. *All eigenvalues of H are integers and we can write*

$$V = V_n \oplus V_{n-2} \oplus \cdots \oplus V_{-n+2} \oplus V_{-n}.$$

Now for any $\mathfrak{sl}_2$-module V (not necessarily irreducible), if $PV = \{v \in V : Xv = 0\}$, then we have the *Lefschetz decomposition of V*:

$$V = PV \oplus YPV \oplus Y^2PV \oplus \cdots .$$

We also know that the maps $Y^m : V_m \to V_{-m}$ and $X^m : V_{-m} \to V_m$ are isomorphisms.

We are close to our goal. We need to figure out how $\mathfrak{sl}_2$ acts on the cohomology ring of an n-dimensional compact Kähler manifold to make it an $\mathfrak{sl}_2$-module. The relevant calculation (conveniently omitted here) is simple but tedious:

$$[\Lambda, L] = n - r$$

on $A^r(M)$. Define

$$h = \sum_{r=0}^{2n} (n - r)\Pi^r.$$

Since $[\Lambda, L] = n - r$ on $A^r(M)$, we know on $A^*(M)$:

$$\begin{aligned} [\Lambda, L] &= \sum_{r=0}^{2n} (n - r)\Pi^r \\ &= h. \end{aligned}$$

We perform similar calculations,

$$\begin{aligned}[h, L] &= \left(\sum_{r=o}^{2n}(n-r)\Pi^r\right) L - L\left(\sum_{r=o}^{2n}(n-r)\Pi^r\right) \\ &= (n-(r+2))\, L - L(n-r) \quad \text{on } A^r(M) \\ &= -2L\end{aligned}$$

and

$$\begin{aligned}[h, \Lambda] &= \left(\sum_{r=o}^{2n}(n-r)\Pi^r\right) \Lambda - \Lambda\left(\sum_{r=o}^{2n}(n-r)\Pi^r\right) \\ &= (n-(r-2))\, \Lambda - \Lambda(n-r) \quad \text{on } A^r(M) \\ &= 2L.\end{aligned}$$

The two calculations above extend linearly to $A^*(M)$. Pulling these calculations together and restating them on $A^*(M)$ for succinctness, we have

$$\begin{aligned}[\Lambda, L] &= h \\ [h, L] &= -2L \\ [h, \Lambda] &= 2\Lambda\end{aligned}$$

Since the operators h, L, Λ commute with Δ_d, they act on the harmonic forms $\mathcal{H}^*(M)$. By the Hodge theorem, $H^*(M) \cong \mathcal{H}^*(M)$, so we can define an action of $\mathfrak{sl}_2$ on the cohomology of M by sending

$$\begin{aligned}X &\to \Lambda \\ Y &\to L \\ H &\to h.\end{aligned}$$

Define the *primitive cohomology* as

$$\begin{aligned}P^{n-k}(M) &= \ker\, L^{k+1} : H^{n-k}(M) \to H^{n+k+2}(M) \\ &= \ker\, \Lambda\, \cap\, H^{n-k}(M).\end{aligned}$$

Together with the facts above about representations of $\mathfrak{sl}_2$, we obtain

Theorem 4.9 (Hard Lefschetz Theorem)**.** *The map*

$$L^k : H^{n-k} \to H^{n+k}$$

is an isomorphism, and we have the Lefschetz decomposition*:*

$$H^m(M) = \bigoplus_k L^k P^{m-2k}(M).$$

So the Kähler condition imposes some strict requirements on the topology of the manifold.

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