Curvature and fundamental group

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1 Introduction

In this paper, we define a growth function on a finitely generated group and we survey some properties on a growth function. And we will prove the two main theorems connecting curvature of a Riemannian manifold and the growth function of its fundamental group. Some examples and further discussion will be given at the end.

2 Growth of finitely generated groups

Let G be a finitely generated group with a specified choice of generators $\{g_1, \cdots, g_p\}$. Then any element g of G can be written as

$$g = \prod_{i=1} g_{k_i}^{r_i} \qquad\qquad (r_i \in \mathbb{Z})$$

with possible repetitions of the generators g_{k_i}.

Such a representation is called a *word* with the respect to the generators, and the integer

$$\sum_i |r_i|$$

is the *length* of that word.

Definition 2.1. *With the notations above, for each positive integer s, a growth function $\gamma(s)$ is the number of distinct group elements which can be expressed as words of length less than s in the specified generators and their inverses.*

Examples

(1) If $G = \langle 1 \rangle$, then $\gamma(s) \equiv 1$.

(2) If $G = \mathbb{Z}_2$, then $\gamma(s) \equiv 2$.

(3) If $G = \langle a \rangle$ with one generator a, then $\gamma(s) \equiv 2s$.

(4) If G is free abelian of rank 2 with specified generators, then $\gamma(s) = 2s^2 + 2s + 1$.

Remarks. Note that there is always an exponential upper bound for $\gamma(s)$. In fact, the inequality

$$\gamma(s+t) \leq \gamma(s)\gamma(t)$$

implies that $\gamma(s) \leq \gamma(1)^s$. With only a little more work, one can verify that the sequence $\gamma(s)^{1/s}$ necessarily converges to a finite limit as $s \to \infty$. For each fixed t, setting $k = [s/t] + 1$, the inequality $\gamma(s) \leq \gamma(kt) \leq \gamma(t)^k \leq \gamma(t)^{1+s/t}$ implies that $limsup\ \gamma(s)^{1/s} \leq \gamma(t)^{1/t}$. Therefore $limsup\ \gamma(s)^{1/s} \leq inf\ \gamma(t)^{1/t} \leq liminf$ $\gamma(s)^{1/s}$, and the sequence converges. It will be convenient to say that a group has *exponential growth* whenever $lim\ \gamma(s)^{1/s} \geq 1$.

Lemma 2.1. *For a free group, which is freely generated by the preferred set of generators* $g_1, \cdots, g_p$, *the growth function is given by*

$$\gamma(s) = (p(2p-1)^s - 1)/(p-1).$$

Hence

$$(2p-1)^s \leq \gamma(s) \leq (2p+1)^s.$$

And for a free abelian group of rank p with generators $\{g_1, \cdots, g_p\}$,

$$\gamma(s) = \sum_{i=1}^{p} 2^i \binom{p}{i} \binom{s}{i}.$$

Proof. Any element of a free group has a unique reduced expression as a word involving the generators g_i's and their inverses. Let $\lambda(s)$ be the number of elements of the group whose reduced expression has length s. Then a reduced word of length s is written as $b_1 b_2 \cdots b_{s-1} b_s$, where the b_i are taken among the generators and their inverses. Since b_{i+1} should be different from b_i^{-1}, then we can get

$$\lambda(s) = (2p)\underbrace{(2p-1)\cdots(2p-1)}_{s-1}, \qquad (s \geq 1).$$

So,

$$\gamma(s) = 1 + \sum_{i=1}^{s} 2p(2p-1)^{s-1} = (p(2p-1)^s - 1)/(p-1)$$

as desired.

Let G be a free abelian group of rank p with generators $g_1, \cdots, g_p$. For a set A, $|A|$ denotes the number of elements of A. Since G is abelian, any element g has a unique reduced expression $g_1^{r_1} \cdots g_p^{r_p}$ i.e g is determined by $r_1, \cdots, r_p$, $r_i \in \mathbb{Z}$. Let $A = \{|a_1| + \cdots + |a_p| \leq s : a_i \in \mathbb{Z}\}$. Then $\gamma(s) = |A|$. It is easy to see that

$$|A| = \sum_{i=0}^{p} 2^i \binom{p}{i} |A_i|$$

where $A_i = \{b_1 + \cdots + b_i \leq s : b_i \in \mathbb{N}\}$. But by the argument of combinations and permutations, we can know that

$$|A_i| = \binom{s}{i}.$$

This completes the proof. □

Lemma 2.2. *Let $\{g_1, \cdots, g_p\}$ and $\{h_1, \cdots, h_q\}$ be two different sets of generators for the same group, and let $\gamma(s)$ and $\gamma'(t)$ be the corresponding growth functions. Then there exist positive constants k and k' so that*

$$\gamma'(t) \leq \gamma(kt)$$

for all t and $\gamma(s) \leq \gamma'(k's)$ for all s.

Proof. If k is large enough so that each h_i can be expressed as a word of length less than k in the generators g_i and g_i^{-1}, then the required inequality $\gamma'(t) \leq \gamma(kt)$ is clearly satisfied. □

3 Growth of Riemannian manifolds

In this section, we state two main theorems.

Theorem 3.1. *If M is a complete n-dimensional Riemannian manifold whose mean curvature tensor R_{ij} is everywhere positive semidefinite, then the growth function $\gamma(s)$ associated with any finitely generated subgroup of the fundamental group $\pi_1(M)$ must satisfy*

$$\gamma(s) \leq constant \cdot s^n.$$

Proof. Choose a base point x_0 in the universal covering space $\widetilde{M}$, and let $V(r)$ denote the volume of the neighborhood $N_r(x_0)$ consisting of all $y \in \widetilde{M}$ with $d(x_0, y) \leq r$, using the Riemannian distance function and the Riemannian n-dimensional volume element in $\widetilde{M}$. If the mean curvature is everywhere nonnegative, then according to $Bishop$[1]:

$$V(r) \leq \omega_n r^n \tag{1}$$

where ω_n denotes the volume of the unit disk in euclidean n-space.

We will identify $\pi_1(M)$ with the group of all covering transformations of $\widetilde{M}$ over M. Let $g_1, \cdots, g_p : \widetilde{M} \to \widetilde{M}$ be the preferred generators for some subgroup of $\pi_1(M)$, and let μ denote the maximum of the numbers $d(x_0, g_i(x_0))$ for $i = 1, 2, \cdots, p$. Note that the neighborhood $N_{\mu s}(x_0)$ contains at least $\gamma(s)$ distinct points of the form $g(x_0)$ with $g \in \pi_1(M)$.

Choose a number $\epsilon > 0$ which is small enough so that the neighborhood $N_\epsilon(x_0)$ in $\tilde{M}$ is disjoint from all of the translates $g(N_\epsilon(x_0))$, with $g \neq 1$. Then the neighborhood $N_{\mu s+\epsilon}(x_0)$ contains at least $\gamma(s)$ disjoint sets of the form $g(N_\epsilon(x_0))$. This proves that

$$\gamma(s)V(\epsilon) \leq V(\mu s + \epsilon). \tag{2}$$

Combining (1) and (2), we obtain the required inequatlity

$$\gamma(s) \leq cs^n, \text{ for } s \geq 1,$$

where $c = \omega_n(\mu + \epsilon)^n / V(\epsilon)$. This proves the Theorem. □

It is conjectured that the group $\pi_1(M)$ itself must be finitely generated. The constant in the equality in the Theorem 3.1 depends on the particular set of generators which is used to define $\gamma(s)$.

Theorem 3.2. *If M is a compact Riemannian manifold with all sectional curvatures less than zero, then the growth function of the fundamental group $\pi_1(M)$ is at least exponetial:*

$$\gamma(s) \geq a^s$$

for some constant $a > 1$.

Proof. Let $-\alpha^2 < 0$ be an upper bound for the sectional curvature of M. Then according to *Günther*[3] the volume $V(r)$ is greater than or equal to the volume

$$n\omega_n \int_0^r (\alpha^{-1} sinh\alpha x)^{n-1} dx$$

of a corresponding ball in the space of constant negative curvature. This expression is asymptotically equal to $2c \cdot exp(\lambda r)$ as $r \to \infty$, where $c = n\omega_n/(n-1)(2\alpha)^n$ and $\lambda = (n-1)\alpha$ are positive constants. Hence

$$V(r) > c \cdot exp(\lambda r) \tag{3}$$

for r sufficiently large.

Let δ be the diameter of M. Let x_0 be a base point in the universal covering space $\widetilde{M}$. Then the projection $\widetilde{M} \to M$ clearly maps the compact neighborhood $N = N_\delta(x_0)$ onto M. Hence the set of all translates gN, with $g \in \pi_1(M)$, forms a covering of $\widetilde{M}$. Note that this covering is locally finite. For otherwise, for some finite r the neighborhood $N_r(x_0)$ would intersect infinitely many of gN, and hence $N_{r+\delta+\epsilon}(x_0)$ would contain infinitely many disjoint sets $gN_\epsilon(x_0)$, each of volume $V(\epsilon) > 0$. This is impossible since $V(r + \delta + \epsilon) < \infty$.

Let F be the finite set consisting of all elements g in $\pi_1(M)$ such that the translate gN intersects N. Let $\mu > 0$ be the minimum of $d(gN, N)$ as gN ranges over all translates of N which do not intersect N.

Claim 1. *If $d(x_0, gN) < \mu t + \delta$ for some positive integer t, then g can be expressed as a t-fold product, $g = f_1 f_2 \cdots f_t$, with $f_1, \cdots, f_t \in F$.*

Proof. Choose $y \in gN$ with $d(x_0, y) < \mu t + \delta$, chosse points $y_1, y_2, \cdots, y_{t+1} = y$ along the minimal geodesic from x_0 to y so that

$$d(x_0, y_1) \leq \delta \text{ and } d(y_i, y_{i+1}) < \mu.$$

Each y_i belongs to some translate $h_i N$, where we can choose h_1 to be 1 and h_{t+1} to be g. Let $f_i = h_i^{-1} h_{i+1}$ so that $f_1 f_2 \cdots f_t = g$. Since the two points $h_i^{-1} y_i$ and $h_i^{-1} y_{i+1}$ have distance less than μ, and belong to N and to $f_i N$ respectively, it follows from the definitions of μ and F that $f_i \in F$. This proves the claim. □

In particular, it follows that the elements of F generate the group $\pi_1(M)$. Let $\gamma'(t)$ denote the growth function of $\pi_1(M)$, computed using F as the set of preferred generators.

According to the claim, the interior of the neighborhood $N_{\mu t+\delta}(x_0)$ is completely covered by the translates $f_1 f_2 \cdots f_t N$ with $f_i \in F$. Since precisely $\gamma'(t)$ of these translates are distinct, this proves that

$$\gamma'(t)V(\delta) \geq V(\mu t + \delta). \tag{4}$$

By Lemma 2.2, there exists a positive integer k so that

$$\gamma(kt) \geq \gamma'(t)$$

or in other words

$$\gamma(s) \geq \gamma'([s/t]), \text{ for } s \geq k. \tag{5}$$

Combining (3),(4) and (5) we clearly obtain

$$\gamma(s) \geq c_1 \cdot exp(\lambda_1 s) \tag{6}$$

for all sufficiently large s, where the constants $c_1 = c \cdot exp(\lambda\delta - \lambda\mu)/V(\delta)$ and $\lambda_1 = \lambda\mu/k$ are both positive. Since $\gamma(s) > 1$ for all $s \geq 1$, it is now easy to choose a constant $a > 1$ so that

$$\gamma(s) \geq a^s$$

for sufficiently large s. This completes the proof. □

Remarks. It may be conjectured that some weaker hypothesis, in which some zero sectional curvatures were allowed, would still be sufficient to imply exponential growth. Perhaps the hypothesis of negative definite mean curvature would already suffice?

4 Examples

1. On a sphere or projective plane or torus or Klein bottle we can impose a metric with nonnegative curvature, so that Theorem 3.1 applies. Choosing the most familiar generators for $\pi_1(M)$, the growth function is given by;

- $\gamma(s) \equiv 1$ for the sphere,
- $\gamma(s) \equiv 2$ for the projective plane P^2,
- $\gamma(s) = 2s^2 + 2s + 1$ for the torus or Klein bottle.

Thus $5s^2$ is an upper bound in each case.

Similarly, since an n-dimensional torus has a free abelian group of rank n as a fundamental group, then Lemma 2.1 implies that the growth function is a polynomial of degree n, providing that the standard generators are used (Compare[4]).

2. Consider the non-orientable surface $P^2\#\cdots\#P^2$ which is obtained from the 2-sphere by attaching p "crossing-caps". If $p \geq 3$, it is known that this surface admits a metric of (constant) negative curvature. The fundamental group has generators, say $x_1, x_2, \cdots, x_p$, subject to the single relation $x_1^2 x_2^2 \cdots x_p^2 = 1$. Note that the elements $x_1, x_2, \cdots, x_{p-1}$ generate a free subgroup. Thus according to Lemma 2.1, we obtain the estimate

$$(2p-3)^s \leq \gamma(s) \leq (2p+1)^s$$

which supports Theorem 3.2.

In the case of an orientable surface of genus $q \geq 2$, a similar argument shows that $(4q-3)^s \leq \gamma(s) \leq (4q+1)^s$.

3. Let G denote the nilpotent Lie group consisting of all 3×3 triangular real matrices with 1's on the diagonal, and let H be the subgroup consisting of all integer matrices of the same form.(In fact, G is called the *Heisenberg* group.) Then the coset space G/H is a compact 3-dimensional manifold with fundamental group H.

Lemma 4.1. *The growth function of H is quartic:*

$$c_1 s^4 \leq \gamma(s) \leq c_2 s^4, \text{ with } c_1, c_2 > 0.$$

Proof. As preferred generators, take the matrices $x = I + e_{23}$ and $y = I + e_{12}$. Then $I + e_{13} = yxy^{-1}x^{-1}$. Let $c = I + e_{13}$. Note that $cx = xc$ and $cy = yc$ i.e c is a central element of H. And any element of H can be written as $x^i y^j c^k$ for $i, j, k \in \mathbb{Z}$. Note that every expression of the form $x^t y^t c^k$ with $0 \leq k \leq t^2$ can be expressed as a word of length $2t$ in x and y. Hence every $x^i y^j c^k$ with $1 \leq i \leq t$, $1 \leq j \leq t$, and $1 \leq k \leq t^2$ has length less than or equal to $4t$, which proves that

$$\gamma(4t) \geq t^4.$$

Conversely, if $x^i y^j c^k$ can be expressed as a word of length less than or equal to s, then it is easily verified that $|i| \leq s, |j| \leq s$, and $|k| \leq (s/2)^2$, which yields a quartic upper bound for $\gamma(s)$. □

Lemma 4.1 shows that $\gamma(s)$ is larger than any constant times s^3, but smaller than any a^s, $a > 1$, for sufficiently large s. Thus we get

Corollary 4.2. *No Riemannian metric on G/H can satisfy either the hypothesis of Theorem 3.1 or the hypothesis of Theorem 3.2.*

If we impose the most natural metric, which comes from a left invariant metric on G, computation shows that the mean curvature tensor of G/H is indefinite, with signature $(+,-,-)$. The scalar curvature ΣR_{ij} turns out to be negative. Thus: *The fundamental group of a compact manifold of negative scalar curvature need not have exponential growth.* On the other hand, the product of a surface of genus 2 with a small sphere provides an example of a manifold of positive scalar curvature whose fundamental group does have exponential growth. Perhaps the scalar curvature of a manifold of dimension ≥ 3 has no influence at all on the fundamental group?

5 Conclusion

We conclude this paper with some remarks. The two main theorems stated in this paper may be contrasted with classical theorems; namely *Myers' Theorem* which asserts that *if mean curvature is positive definite on a compact manifold, then the fundamental group is finite*; the *Hadamard − Cartan Theorem* which asserts that *if all sectional curvatures are nonpositive on a complete manifold, then the higher homotopy groups $\pi_i(M)$, $i \geq 2$, are zero*; and *Preissmann's Theorem* which asserts that *if all sectional curvatures are negative on a compact manifold, then every abelian subgroup of $\pi_1(M)$ is cyclic.*

Finally, for $n \geq 3$, there are n-dimensional compact manifolds whose fundamental group has exponential growth and which carry no metric with negative sectional curvature ([8]). This implies that the converse of Theorem 3.2 is not true in general.

References

[1] R. L. Bishop, *A relation between volume, mean curvature, and diameter,* Notices Amer. Math. Soc. 10 (1963) 364.

[2] —& R. J. Crittenden, *Geometry of manifolds,* Academic Press, New York, 1964.

[3] P. Günther, *Eininge Sätze über das Volumenelement eines Riemannschen Raumes,* Publ. Math. Debrecen 7 (1960) 78-93.

[4] H. U. Krause, *Gruppensruktur und Gruppenbild,* Thesis, Eidg. Techn. Hochs. Zürich 1953, 45pp.

[5] S. Myers, *Riemannian manifolds with positive mean curvature,* Duke Math. J. 8 (1941) 401-404.

[6] A. Preissmann, *Quelques propriétés globales des espaces de Riemann,* Comment. Math. Helv. 15 (1942-43) 175-216.

[7] J.L. Synge, *On the connectivity of spaces of positive curvature,* Quart. J. Math. Oxford, Ser. (1), 7 (1936) 316-320.

[8] *Variétés à courbure négative,* Publications Mathématiques de l'université Paris 7.