

QUASI-CONVEXITY AND SHRINKWRAPPING

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ABSTRACT. We extend a result of Minsky to show that for a map of a surface to a hyperbolic 3-manifold which is not necessarily π_1 -injective but is 2-incompressible rel a geodesic link with a definite tube radius, the set of non-contractible simple loops with a bounded length representatives is quasi-convex in the complex of curves.

1. INTRODUCTION

Suppose S is a closed orientable surface of genus ≥ 2 . The complex of curves of S , denoted by $\mathcal{C}(S)$, is a complex whose vertices are homotopy classes of non-contractible simple loops on S . Two vertices are connected by an edge if they have disjoint representatives on the surface. Using this we endow $\mathcal{C}(S)$ by a geodesic metric where the length of each edge is one. One of our motivation in this article is the following question:

Question. Suppose $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{C})$ is a representation of $\pi_1(S)$ and $B \geq 0$. Let $\mathcal{C}(B, j) \subset \mathcal{C}(S)$ denotes the homotopy classes of simple loops α on S , where the translation length of $\rho(\alpha)$ is at most B . Is $\mathcal{C}(B, j)$ a quasi-convex subset of $\mathcal{C}(S)$?

Recall that for a geodesic metric space X , a subset $Y \subset X$ is K -quasi-convex if for every two points $p, q \in Y$, every geodesic connecting p and q is contained in the K -neighborhood of Y . Also note that even though a closed curve α only identifies a conjugacy class in $\pi_1(S)$, it makes sense to speak of the translation length of $\rho(\alpha)$ because the translation length stays invariant under conjugacy.

Our result lies somewhere between two partial answers to the above question. These results have been significant in relating the topology and geometry of 3-manifolds to the combinatorics of curves on a surface. The first result is a theorem of Minsky [Min01] which has been a central theme in the construction of models for hyperbolic 3-manifolds and the proof of Thurston's Ending Lamination Conjecture [ELC1, ELC2]. Suppose $j : S \rightarrow N$ is a π_1 -injective map from a closed surface S of genus > 1 into the hyperbolic 3-manifold N and assume $\mathcal{C}(B, j)$ denotes the homotopy classes of non-contractible simple loops α where $j(\alpha)$ has a representative of length $\leq B$ in N . For B large enough depending on $\chi(S)$, the set $\mathcal{C}(j, B)$ is quasi-convex in the complex of curves of S with a quasi-convexity constant that depends only on $\chi(S)$ and B .

The other result is more topologic and considers the case when M is a compact 3-manifold and S is a compressible component of ∂M . Then Masur and Minsky [MM03] show that the homotopy classes of non-contractible simple loops on S which are compressible in M form a quasi-convex subset of the complex of curves of S . Here the quasi-convexity constant depends only on $\chi(S)$.

The above question has been motivated mostly by the first result and because of the major role of Minsky's result [Min01] in the study and description of the geometry of hyperbolic 3-manifolds. Complete or partial answers to the question then, become important steps to understand how much of those results can possibly be carried out to a wider setting.

We extend Minsky's result in [Min01] by replacing the condition that j is π_1 -injective with a weaker condition. Given a hyperbolic 3-manifold N , we say a geodesic link $\Gamma \subset N$ is η -separated for $\eta > 0$ if Γ has an embedded tubular neighborhood of radius η . Suppose N has no parabolics and $j : S \rightarrow N$ is 2 -incompressible with respect to an η -separated geodesic link $\Gamma \subset N$. By 2 -incompressibility of j , we mean it is π_1 -injective as a map to $N \setminus \Gamma$ and the image of a loop cannot be freely homotopic to a (multiplied) meridian of a component of Γ . Finally for a loop γ on S , a *representative* in $N \setminus \Gamma$ is a closed curve that is freely homotopic to $j(\gamma)$ in $N \setminus \Gamma$. The set $\mathcal{C}(B, j) \subset \mathcal{C}(S)$ consists of non-contractible simple loops with a representative of length $\leq B$ in $N \setminus \Gamma$.

Theorem 1.1. *Suppose N is an orientable hyperbolic 3-manifold, $\Gamma \subset N$ is an η -separated geodesic link and $j : S \rightarrow N$ is 2 -incompressible rel Γ . Then for B sufficiently large, the set $\mathcal{C}(B, j)$ is K -quasi-convex for a constant K that depends only on $\chi(S), \eta$ and B .*

This is obviously stronger than Minsky's result. One advantage of our proof is that the constant K is made more effective and can compute K as a function of $\chi(S), \eta$ and B .

In applications, one needs to guarantee existence of η -separated geodesic links which make a map $j : S \rightarrow N$ 2 -incompressible. One way of obtaining such links is by using "wide product regions." We show how one can obtain such geodesic links by using product regions. Suppose $j : S \rightarrow N$ is an embedding of an oriented closed surface of genus ≥ 2 to the oriented hyperbolic 3-manifold N . The surface $j(S)$ is 2 -sided and by choosing orientations on S and N , we can denote one side as the positive side and the other as the negative side. We assume a small constant $\epsilon_0 > 0$ is fixed. A *product region* is a submanifold of N homeomorphic to $S \times [0, 1]$. We say $j(S)$ is *blocked from the positive (resp. negative) side* by a product region U , if U is disjoint from $j(S)$ and there is a homotopy between $j(S)$ and a component of ∂U that stays on the positive (resp. negative) side of $j(S)$. In section 5, we precisely define the "width" of a block which depends on ϵ_0 and another constant $L > 0$ that measures the distortion from a "standard" product region. In particular when $j(S)$ is incompressible to the positive (resp. negative) side, we say it is blocked from the positive (resp. negative) side by the empty set. Then the empty set is the block and we use the convention to say it has infinite width.

Theorem 1.2. *Suppose $\epsilon_0, L > 0$ are fixed and $j : S \rightarrow N$ is a proper embedding of the oriented surface S with $\chi(S) < 0$ into an oriented hyperbolic 3-manifold N . Suppose $j(S)$ is blocked from positive and negative sides by blocks U^+ and U^- (possibly empty sets) of sufficiently large “width”. Then there exists a 0.025 -separated geodesic link $\Gamma \subset U^+ \cup U^-$ such that $j(S)$ is 2-incompressible rel Γ .*

Again, we explicitly compute an upper bound for the required “width” of the blocks. Note that in the introduction, we have discussed only the cases when S is closed and N has no parabolics but we will remove these assumptions when we prove the theorems.

The main application of the above theorems is in construction of models and interpolations. This is obviously an extension of similar application for Minsky’s quasi-convexity results in [Min01] and [ELC1]. The advantage of our approach is to find local models by using weaker assumptions. In [Na] our results here are used to obtain models for a class of hyperbolic structures on handlebodies and then it is used to construct nearly hyperbolic negatively curved metrics on a class of closed 3-manifolds. Also in joint work with Brock, Minsky and Souto [BMNS], this result is used to produce models that describe certain regions in 3-manifolds. In all these results, one is dealing with a map of a surface to a hyperbolic 3-manifold which is not π_1 -injective, for example when S is a Heegaard surface. But using other features of the geometry, one can find η -separated geodesic links which make the surface 2-incompressible. This is usually by constructing wide product regions and applying Theorem 1.2. Then Theorem 1.1 can be used successfully.

A main ingredient of the proofs is a use of *shrinkwrapped* surfaces. The original construction of such maps was developed in work of Calegari-Gabai in the proof of the Tameness Conjecture. We, however, rely more on softer constructions of such surfaces which were obtained in work of Soma [So06] and are called CAT(−1)-*piecewise ruled maps*. We explain the construction of these surfaces and the main properties of them in the presence of η -separated geodesic links in section 3. Then in section 4, we use these surfaces to prove a generalization of Theorem 1.1. We basically reconstruct Minsky’s argument in [Min01] by using shrinkwrapped surfaces. We define a projection $\Pi_{j,B}$ from $\mathcal{C}(S)$ to $\mathcal{C}(B, j)$ and in Proposition 4.4, we prove it satisfies two important properties. Then a standard argument of Minsky proves the image $\mathcal{C}(B, j)$ is quasi-convex. We should point out that in the proof of these properties, we prove a weak Uniform Injectivity Theorem for the shrinkwrapped surfaces. Our proof is an elementary case of the proof of the Uniform Injectivity Theorem given in [NS]. This is one of the main differences with Minsky’s proof and the constructive nature of this proof allows a computation of an upper bound for the quasi-convexity constant.

In section 5, we prove a generalized version of Theorem 1.2. We use the wide product region to find many geodesic closed curves with large pairwise distance. Then we use an argument of Gabai [Ga97] to deform these in a way that the length decreases and eventually, they have a definite tube radius. Now we prove by means of contradiction, if $j : S \rightarrow N$ is not 2-incompressible, then we allow compressions that each of them intersects Γ at most once. When we are done, the obtained surfaces S_1, \dots, S_k are 2-incompressible with respect to the subset Γ_1 of Γ that has

not been involved in the process. When the number of components of Γ is large enough then the number of components of Γ_1 is also large and we use this to show for one of the surfaces S_i , we must have $\chi(S_i) \leq \chi(S)$ which is a contradiction. This argument owes a huge debt to the argument of Calegari-Gabai in [CG04] and follows their ideas.

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2. PRELIMINARIES

2.1. Hyperbolic geometry. A *hyperbolic manifold* is a complete Riemannian manifold with finitely generated fundamental group and constant sectional curvature -1 . Recall that in a Riemannian manifold N , the *injectivity radius* of N , denoted by $\text{inj}_N(x)$, at a point $x \in N$ is half the length of the shortest (homotopically nontrivial) loop through x . We use the notation $N^{\leq \epsilon}$ (resp. $N^{< \epsilon}, N^{\geq \epsilon}, N^{> \epsilon}$) to denote the set of points $x \in N$ where $\text{inj}_N(x) \leq \epsilon$ (resp. $\text{inj}_N(x) < \epsilon, \text{inj}_N(x) \geq \epsilon, \text{inj}_N(x) > \epsilon$). By Margulis lemma, there exists a universal constant $\epsilon_M > 0$, such that in a hyperbolic 3-manifold N and for every $\epsilon \leq \epsilon_M$, every component of $N^{\leq \epsilon}$, the ϵ -thin part of N , is either

- (1) a rank two cusp: a horoball in \mathbb{H}^3 modulo a parabolic action of $\mathbb{Z} \oplus \mathbb{Z}$,
- (2) a rank one cusp: a horoball in \mathbb{H}^3 modulo a parabolic action of \mathbb{Z} , or
- (3) a solid torus neighborhood of a closed geodesic.

The components of type (1) and (2) above are called *cusps* of N and a component T of type (3) is called a *Margulis tube*. A Margulis tube T is a solid torus neighborhood of a closed geodesic γ_T of length $\leq \epsilon$, which we call the *core of the Margulis tube*. The following lemma gives an upper bound for the distances of nested components of the thin part.

Lemma 2.1. *There exists a homeomorphism $\xi : [1, \infty) \rightarrow [0, \infty)$ such that for every hyperbolic 3-manifold N and $\epsilon_2 < \epsilon_1$ both smaller than the Margulis constant*

$$d_N(N^{\leq \epsilon_2}, N^{\geq \epsilon_1}) \geq \xi(\epsilon_1/\epsilon_2).$$

In particular, using the above lemma, we assume the Margulis constant ϵ_M is chosen such that the distance between every two distinct components of $N^{\leq \epsilon_M}$ is at least 1. Given $x, y \in N$ and $\epsilon \leq \epsilon_M$, we define

$$d_N^{\geq \epsilon}(x, y) := l_N([x, y]_N \cap N^{\geq \epsilon})$$

where $[x, y]_N$ denotes the geodesic that connects x and y in N and $l_N([x, y]_N \cap N^{\geq \epsilon})$ is the total lengths of components of $[x, y]_N \cap N^{\geq \epsilon}$. Similarly for $A \subset N$ we define $\text{diam}_N^{\geq \epsilon}(A)$ to be the supremum of $d_N^{\geq \epsilon}(x, y)$ for points $x, y \in A$ and for two subsets $A_1, A_2 \subset N$, $d_N^{\geq \epsilon}(A_1, A_2)$ denotes the infimum of $d_N^{\geq \epsilon}(x, y)$ for points $x \in A_1, y \in A_2$.

We will need the following lemma in our arguments.

Lemma 2.2. *Let N be a hyperbolic 3-manifold and α a homotopically non-trivial closed curve in N and α^* its geodesic representative. Then*

$$\cosh d_N(\alpha, \alpha^*) \leq l_N(\alpha)/l_N(\alpha^*),$$

where $l_N(\alpha)$ is the length of α as a curve in N . Also given $\epsilon > 0$

$$\cosh d_N^{\leq \epsilon}(\alpha, \alpha^*) \leq l_N(\alpha)/\epsilon,$$

where $d_N^{\leq \epsilon}$ denotes the distance rel $N^{\leq \epsilon}$.

A complete hyperbolic metric on the interior of a compact 3-manifold M canonically corresponds the conjugacy class of a discrete and faithful representation $\rho : \pi_1(M) \rightarrow \text{Isom}(\mathbb{H}^3)$. The image $\rho(\pi_1(M))$ of this representation is a Kleinian group and we can consider the limit set of its action which is a closed subset Λ of the sphere at infinity. The convex hull of Λ is invariant under the action of $\rho(\pi_1(M))$ and its projection to the hyperbolic 3-manifold is called the *convex core*. When $M = S \times [0, 1]$ and the limit set of $\rho(\pi_1(M))$ is the entire sphere at infinity, we say the hyperbolic structure is *doubly degenerate*.

2.2. Complex of Curves. Suppose $\bar{S} = S_{g,b}$ is a compact surface of finite type with genus g and b boundary components. We also always assume $\chi(\bar{S}) < 0$. A *peripheral* closed curve γ on \bar{S} is a closed curve that is homotopic to a multiple of a component of $\partial\bar{S}$ in \bar{S} . A closed curve is *non-peripheral* if it is non-contractible and is not freely homotopic to a multiple of a peripheral closed curve on \bar{S} . Also if α is a properly embedded arc in \bar{S} , we say α is non-peripheral if α is not homotopic rel $\partial\bar{S}$ to an arc contained in $\partial\bar{S}$.

The *complex of curves* $\mathcal{C}(S)$ of \bar{S} is a complex where every vertex is either a homotopy class of a non-peripheral simple loop or is the homotopy class (rel $\partial\bar{S}$) of a non-peripheral properly embedded arc. A subset $\{v_0, \dots, v_n\}$ of distinct vertices form an n -simplex if these homotopy classes have representatives on \bar{S} which do not intersect. We equip $\mathcal{C}_1(S)$, the one-skeleton of $\mathcal{C}(S)$, with a path metric such that the length of every edge is equal to one. It is not hard to extend this metric to $\mathcal{C}(S)$ and in fact most of the time we treat $\mathcal{C}(S)$ without distinguishing it from its 1-skeleton.

Masur and Minsky [MM99] proved that if \bar{S} is not the 3-holed sphere $S_{0,3}$ then $\mathcal{C}(S)$ has infinite diameter and is hyperbolic in sense of Gromov.

3. SHRINKWRAPPING

A *geodesic link* Γ in a hyperbolic 3-manifold N is a disjoint union of finitely many simple closed geodesics in N . A Γ -*homotopy* is a homotopy $F : X \times [0, 1] \rightarrow N$ such that if $F(x, 0) \notin \Gamma$ then $f(x, t) \notin \Gamma$ for $t < 1$ and if $f(x, 0) \in \Gamma$ then $f(x, t) = f(x, 0)$ for all $t \in [0, 1]$. We say $X_1 = F(X \times \{1\})$ is Γ -*homotopic* to $X_0 = F(X \times \{0\})$.

When Γ is a nonempty geodesic link in a hyperbolic 3-manifold N , $\widetilde{N \setminus \Gamma}$ is an incomplete hyperbolic 3-manifold. We consider the universal cover $N \setminus \Gamma$ and its metric completion. The action of $\pi_1(N \setminus \Gamma)$, via deck transformations, extends to an

action on the completion. We denote the quotient by $\overline{N \setminus \Gamma}$. We can visualize the construction as an infinite branching $\overline{N \setminus \Gamma} \rightarrow N$ with Γ the branching locus. This is the identity on $N \setminus \Gamma$ and using this projection, we speak of pre-images of subsets of N in $\overline{N \setminus \Gamma}$. In this setting Soma [So06] showed that the metric completion of $\overline{N \setminus \Gamma}$ is CAT(-1). Then the following lemma follows from the general theory of CAT(κ) metrics.

Lemma 3.1. [Ga, So06] *Let $x, y \in N$. If $\alpha_0 \subset N \setminus \Gamma$ is a path from x to y or a loop disjoint from Γ , then there exists a unique piecewise geodesic α_1 both path and Γ -homotopic to α_0 such that α_1 is length minimizing among all paths both path and Γ -homotopic to α_0 . The function that takes α_0 to α_1 is continuous.*

A piecewise geodesic α_1 obtained in the conclusion of the above lemma is called a Γ -geodesic. It is not hard to generalize the above lemma by assuming that one of the points x, y or both of them are contained on $\partial\mathbb{H}^3$ where $\partial\mathbb{H}^3$ is considered as the boundary at infinity of the universal cover of N . We consider a point of $\partial\mathbb{H}^3$ as an ideal point and it makes sense to speak of paths that connect x and y . Then an easy generalization of the above lemma shows that every such path α_0 is Γ -homotopic to a path α_1 which is locally a Γ -geodesic.

As before we assume S is the interior of a compact surface \overline{S} of finite type. A proper continuous map $j : S \rightarrow N$ is said to be *2-incompressible* in N rel a geodesic link Γ if

- $j(S) \cap \Gamma = \emptyset$ and $j_* : \pi_1(S) \rightarrow \pi_1(N \setminus \Gamma)$ is injective,
- the j -image of a non-contractible loop is not freely homotopic in $N \setminus \Gamma$ to a (multiplied) meridian of any component of Γ and
- given a neighborhood U of a cusp of N such that $U \cap \Gamma = \emptyset$, the j -image of a non-peripheral simple loop is not freely homotopic in $N \setminus \Gamma$ into U .

Suppose $\Gamma \subset N$ is a geodesic link and $j : S \rightarrow N$ is a 2-incompressible map rel Γ . For a closed curve α we say $\alpha^* \subset N$ is the Γ -geodesic representative of α if α^* is the unique Γ -geodesic that is both path and Γ -homotopic to $j(\alpha)$. It follows from the definition of 2-incompressible maps that every non-peripheral simple loop has a unique Γ -geodesic representative.

When S is the interior of a finite type surface, by a triangulation of S we mean a finite triangulation where there is a vertex at infinity for every cups of S . More precisely assume S is equipped with a finite area hyperbolic metric, identify the universal cover of S by \mathbb{H}^2 and consider the triangulation on the compactification of \mathbb{H}^2 by adding the circle at infinity. Then we consider $\pi_1(S)$ -equivariant triangulations where the fixed points of the parabolic elements of $\pi_1(S)$ are all the ideal points of the triangulation and the projection of the triangulation to S has a finite number of cells.

Definition 3.1. Let $f : S \rightarrow \overline{N \setminus \Gamma}$ be a continuous map from an orientable surface S . Suppose S admits a triangulation K . We say that f is a *piecewise ruled map* with respect to K if (i) for each edge e of K , $f(e)$ is a broken line consisting of finitely many hyperbolic segments, and (ii) for each 2-cell F of K , $f(F)$ is a ruled

triangle based on a single vertex. This map f is called a CAT(-1)-*piecewise ruled map rel Γ* if the cone-angle of S at any singular point is at least 2π .

When the choice of Γ is clear, we sometimes simply say f is a CAT(-1)-piecewise ruled map. We say a map $f : S \rightarrow N$ is a piecewise ruled map if it factors as the composition of a piecewise ruled map $g : S \rightarrow \overline{N \setminus \Gamma}$, and the canonical projection $\overline{N \setminus \Gamma} \rightarrow N$. In this article, we are only interested in CAT(-1)-piecewise ruled maps which are obtained by a Γ -homotopy on a 2-incompressible map $j : S \rightarrow N \setminus \Gamma$. In particular in all our proofs it is implicit that with a small deformation, we can push the surface away from the geodesic link.

Suppose $j : S \rightarrow N \setminus \Gamma$ is 2-incompressible rel Γ . Given a multi-curve γ on S , we say a CAT(-1)-piecewise ruled map $f : S \rightarrow N$ *realizes γ rel Γ* if there is a Γ -homotopy from j to f and the image of γ by f is a collection of Γ -geodesics. The following theorem is essentially due to Soma.

Theorem 3.2. [So06] *Suppose $\Gamma \subset N$ is a geodesic link in the orientable hyperbolic 3-manifold N and $j : S \rightarrow N$ is a 2-incompressible map rel Γ of a surface S with $\chi(S) < 0$. Then there exists a Γ -homotopy $F : S \times [0, 1] \rightarrow N$ such that $F(x, 0) = j(x)$ and $F(x, 1)$ is a CAT(-1)-piecewise ruled map. Furthermore if γ is a multi-curve whose components are non-peripheral, we can assume that $F(x, 1)$ realizes γ rel Γ .*

Proof. We assume S is equipped with a finite area hyperbolic metric and therefore every component of \overline{S} corresponds to a cusp. Let $\alpha_1, \dots, \alpha_k$ be pairwise disjoint non-peripheral simple loops in S which define a pants decomposition of S . If a multi-curve γ is given, assume $\gamma \subset \{\alpha_1, \dots, \alpha_k\}$.

Consider a triangulation of K of S such that there is an ideal vertex associated to every cusp and all other vertices are contained in $\alpha_1 \cup \dots \cup \alpha_k$. Every curve α_i , $i = 1, \dots, k$, has a unique Γ -geodesic representative in N . After deforming j by a Γ -homotopy, we can assume each $j(\alpha_i)$ is a closed Γ -geodesic. We use another Γ -homotopy to map every ideal vertex associated to a cusp of S to the associated cusp of N and map $j(e)$ to a Γ -geodesic segment for any edge e of K . For any 2-dimensional face F of K , take a vertex v_0 and the opposite edge e_0 . Then $j|_F$ can be homotoped rel ∂F and via a Γ -homotopy to a map $g|_F$ such that $g(F)$ is a ruled triangle consisting of all Γ -geodesic segments connecting $j(v_0)$ with points of $j(e_0)$. Putting such maps together defines a piecewise ruled map $g : S \rightarrow N$ which is Γ -homotopic to j . From the construction of g , for any singular point $g(v)$ of $g(S)$, there exists an arc κ in S with $v \in \text{Interior}(\kappa)$ and such that $g(\kappa)$ is a Γ -geodesic. An argument in [So06] applies wordy by word here to show that the cone angle of S at every such point is at least 2π . This shows g is a CAT(-1)-piecewise ruled map. \square

With an abuse of notation, we say the CAT(-1)-piecewise ruled map f is *2-incompressible rel Γ* if it is Γ -homotopic to a map $j : S \rightarrow N \setminus \Gamma$ which is 2-incompressible rel Γ , i.e. there exists a small deformation of f that produces a 2-incompressible map.

Simplicial hyperbolic surfaces are special classes of $\text{CAT}(-1)$ -piecewise ruled maps. Here we basically define a simplicial hyperbolic surface to be a $\text{CAT}(-1)$ -piecewise ruled map in $\overline{N \setminus \Gamma}$ whose image does not intersect Γ .

Definition 3.2. We say a geodesic link Γ in N is η -separated if any path $\alpha : I \rightarrow N$ with endpoints on Γ satisfying

$$l(\alpha(I)) \leq \eta$$

is homotopic (rel endpoints) into Γ .

Remark 3.3. In what follows and in practice, the geodesic links that are constructed have a bounded total length, which may depend on the topology of the surface; but we do not need such an assumption in our statements.

One important property of an η -separated geodesic link Γ , which will be used later, is that Γ cannot enter thin parts of N unless it has a component which is the geodesic core of a Margulis tube. We always assume η is smaller than the Margulis constant. In this case, more precisely if T is a component of $N^{\leq \eta/2}$ then $T \cap \Gamma$ is either empty or T is a Margulis tube with the geodesic core γ^* and $\gamma^* \in \Gamma$ is a component of Γ .

Lemma 3.3. *Suppose $\alpha : S^1 \rightarrow N$ is a piecewise linear or smooth map to a hyperbolic manifold N which is homotopically trivial in N . Then there exists a map $g : D^2 \rightarrow N$ of the 2-disk D^2 whose restriction to $S^1 = \partial D^2$ is α and the diameter of $g(D^2)$ in N is bounded from above by the length of α in N .*

The proof of the above lemma is a simple construction of a ruled disk by taking a point of the curve and use it to cone off all other points. This disk immediately has the required property. Using this we can prove the following.

Lemma 3.4. *Suppose $f : S \rightarrow N$ is a $\text{CAT}(-1)$ -piecewise ruled map which is 2-incompressible rel Γ , an η -separated geodesic link in the hyperbolic 3-manifold N . For every non-contractible simple loop α on S with $\text{length}(f(\alpha)) \leq \eta$, the loop $f(\alpha)$ is homotopically nontrivial in N . In particular*

$$f(\Sigma_f^{\leq \epsilon}) \subset N^{\leq \epsilon}$$

where Σ_f denotes the surface S equipped with the path metric induced from f and $\epsilon \leq \frac{\eta}{2}$.

Proof. Suppose a non-contractible simple loop α in S has length $\leq \eta$ in Σ_f or equivalently length of $f(\alpha)$ is at most η . By lemma 3.3, if $f(\alpha)$ is homotopically trivial in N there exists a map $g : D^2 \rightarrow N$ where $g|_{\partial D^2} = f|_{\alpha}$ and its diameter is $\leq \eta$. Because of the 2-incompressibility of f the disk $g(D^2)$ has to intersect Γ at least twice. Besides there are two of those points $g(x), g(y) \in \Gamma$ such that the g -image of the arc $[x, y]$ that connects x and y in D^2 is not homotopic (rel endpoints) into Γ . So by η separability of Γ the distance between $g(x)$ and $g(y)$ is $> \eta$ and this contradicts the bound for the diameter of $g(D^2)$. The second statement of the lemma follows immediately. \square

One immediate corollary of this lemma is a Bounded Diameter Lemma for such surfaces

Lemma 3.5 (Bounded Diameter Lemma). *Suppose Γ is an η -separated geodesic link in the hyperbolic 3-manifold N and $f : S \rightarrow N$ is a $\text{CAT}(-1)$ -piecewise ruled map which is 2-incompressible rel Γ . Then for every $\epsilon > 0$,*

$$\text{diam}_N^{\geq \epsilon}(f(S)) \leq D$$

where D depends only on ϵ, η and $\chi(S)$.

Proof. Without loss of generality assume $\epsilon \leq \frac{\eta}{2}$. By lemma 3.4 we know

$$f(\Sigma_f^{\leq \epsilon}) \subset N^{\leq \epsilon}$$

where Σ_f is the surface S equipped with the path metric induced from f . Since f is a piecewise ruled map, Σ_f has curvature ≤ -1 at every point and in particular using the Gauss-Bonnet theorem, we see that the area of Σ_f is at most $-2\pi\chi(S)$. A standard argument then shows that every two points $x, y \in \Sigma_f^{\geq \epsilon}$ are connected by a geodesic whose intersection with $\Sigma_f^{\geq \epsilon}$ has length $\leq -2\pi\chi(S)/\epsilon$. So obviously the points $f(x)$ and $f(y)$ can be connected with a path whose intersection with $N^{\geq \epsilon}$ has length $\leq -2\pi\chi(S)/\epsilon$ and this proves the lemma. \square

For $\text{CAT}(-1)$ -piecewise ruled maps, we need something stronger than the Bounded Diameter Lemma to tell us that the map f respects the thick-thin decompositions of Σ_f and N . We prove this in proposition 3.8 but first we need to understand thin parts of $\overline{N \setminus \Gamma}$ and Σ_f . Using the same notation as the one for complete hyperbolic manifolds, we define $\overline{N \setminus \Gamma}^{\leq \epsilon}$ to denote the set of points $x \in \overline{N \setminus \Gamma}$ where the injectivity radius at x is at most ϵ . Recall that we always assume η is smaller than the Margulis constant.

Lemma 3.6. *Given $\epsilon \leq \eta/8$, a component of $\overline{N \setminus \Gamma}^{\leq \epsilon}$ is either pre-image of a component of $N^{\leq \epsilon}$ or is pre-image of a component of $\mathcal{N}_{2\epsilon}(\Gamma)$, where $\mathcal{N}_{2\epsilon}(\Gamma)$ denotes the 2ϵ tubular neighborhood of Γ . In particular, each component of $\overline{N \setminus \Gamma}^{\leq \epsilon}$ has abelian fundamental group.*

Proof. Assume $x \in \overline{N \setminus \Gamma}^{\leq \epsilon}$ then there exists a loop γ_x based at x of length $\leq 2\epsilon$. The image of this loop by the branching map is a loop of the same length in N . If it is homotopically nontrivial then it is contained in a component T of $N^{\leq \epsilon}$. Because of η -separability and since $\epsilon \leq \eta/8$, every component of Γ either has distance $> \eta/4 \geq 2\epsilon$ from T or is the core geodesic of T . If the projection of γ_x to N is homotopically trivial then by lemma 3.3, it bounds a disk of diameter $\leq 2\epsilon$ and Γ intersects this disk. Hence it is contained in the 2ϵ -neighborhood of Γ and therefore x belongs to the pre-image of a component of $\mathcal{N}_{2\epsilon}(\Gamma)$. Even more this argument shows that if length of a component of Γ is $> 2\epsilon$ then its 2ϵ -neighborhood does not intersect the pre-image of any component of $N^{\leq \epsilon}$ which finishes the proof of the lemma. \square

Lemma 3.7. *Suppose Γ is an η -separated geodesic link in a hyperbolic 3-manifold N and $f : S \rightarrow N$ is a $\text{CAT}(-1)$ -piecewise ruled map that is 2-incompressible rel Γ . Also let Σ_f denotes the surface S equipped with the path metric induced by f . Then for any $\epsilon \leq \eta/8$, every component of $\Sigma_f^{\leq \epsilon}$ is either an annular neighborhood*

of a simple closed geodesic of length $\leq \epsilon$ or is an annular neighborhood of a cusp. Furthermore $\Sigma_f^{\geq \epsilon}$ has no annular or disk components.

Proof. By lemma 3.4, $f(\Sigma_f^{\leq \epsilon}) \subset N^{\leq \epsilon}$ for every $\epsilon < \eta/2$. Obviously we can view this map as a map whose target is $N \setminus \Gamma$; we denote this induced map by $\bar{f} : S \rightarrow \overline{N \setminus \Gamma}$. This map is π_1 -injective and the image of a component V of $\Sigma_f^{\leq \epsilon}$ is contained in \bar{T} , the pre-image of a component T of $N^{\leq \epsilon}$, which is also a component of the ϵ -thin part of $\overline{N \setminus \Gamma}$.

Let $\tilde{\Sigma}_f$ denotes the universal cover of Σ_f equipped with the lift of the metric on Σ_f . Suppose \tilde{V} denotes a lift of V to $\tilde{\Sigma}_f$ and $\tilde{x} \in \tilde{V}$. Then there exists a primitive element $h_{\tilde{x}} \in \pi_1(\Sigma_f)$ such that $d_{\tilde{\Sigma}_f}(\tilde{x}, h_{\tilde{x}}(\tilde{x})) \leq \epsilon$, where $h_{\tilde{x}}$ acts on $\tilde{\Sigma}_f$ as a deck transformation. The map \bar{f} on the level of fundamental groups takes $h_{\tilde{x}}$ to an element of $\pi_1(\bar{T})$. As we mentioned $\pi_1(\bar{T})$ is abelian and because of the π_1 -injectivity of the map \bar{f} , we must have $h_{\tilde{x}} = h_{\tilde{y}}$ or $h_{\tilde{x}} = h_{\tilde{y}}^{-1}$ for every $\tilde{x}, \tilde{y} \in \tilde{V}$. (Recall that $h_{\tilde{x}}$ and $h_{\tilde{y}}$ are chosen to be primitive.) This basically shows that we can choose $h = h_{\tilde{x}}$ independent of $\tilde{x} \in \tilde{V}$ and this proves \tilde{V} is contained in

$$\{\tilde{x} \in \tilde{\Sigma}_f : d_{\tilde{\Sigma}_f}(\tilde{x}, h(\tilde{x})) \leq \epsilon\}.$$

Obviously the image of every point in the above set must also belong to $\Sigma_f^{\leq \epsilon}$ and therefore it is equal to \tilde{V} . Note in addition that \tilde{V} is convex because $d(\tilde{x}, h(\tilde{x}))$ is a convex function of \tilde{x} .

Now we claim that V , the projection of \tilde{V} , is the same as $\tilde{V}/\langle h \rangle$, the quotient of \tilde{V} by the action of h , and therefore is a non-contractible annulus. This follows if we show that for every $g \in \pi_1(\Sigma_f)$, $g\tilde{V} \cap \tilde{V} = \emptyset$ unless g is a power of h . But if $g\tilde{V} \cap \tilde{V} \neq \emptyset$ then because of the convexity of \tilde{V} , we can see that the induced map on the fundamental groups, takes g to an element of $\pi_1(\bar{T})$. Again by using the fact that $\pi_1(\bar{T})$ is abelian and \bar{f} is π_1 -injective, it follows that g must be a power of h . This proves the claim and as a corollary, we see that V is an essential annulus in Σ_f .

So obviously $\Sigma_f^{\geq \epsilon}$ has no disk components. Two parallel annuli cannot be components of $\Sigma_f^{\leq \epsilon}$ because then it means that we can lift them to disjoint sets \tilde{V}_1, \tilde{V}_2 in $\tilde{\Sigma}_f$ and find a primitive element $h \in \pi_1(\Sigma_f)$ such that

$$V_1 = \tilde{V}_1/\langle h \rangle \quad \text{and} \quad V_2 = \tilde{V}_2/\langle h \rangle.$$

But then our characterization above shows that we must have

$$\tilde{V}_1 = \tilde{V}_2 = \{\tilde{x} \in \tilde{\Sigma}_f : d_{\tilde{\Sigma}_f}(\tilde{x}, h(\tilde{x})) \leq \epsilon\},$$

and $V_1 = V_2$ cannot be disjoint.

An annular neighborhood of every cups of Σ_f is contained in $\Sigma_f^{\leq \epsilon}$ so the above argument repeated when h is a parabolic element of $\pi_1(\Sigma_f)$ shows that $\Sigma_f^{\leq \epsilon}$ does not contain a component which is a boundary parallel compact annulus. These two facts together show that $\Sigma_f^{\geq \epsilon}$ has no annular component either. \square

Now using our observations above, we can show that in some sense f respects the thick-thin decomposition

Proposition 3.8 (Thick Goes to Thick). *Given $\eta > 0$ smaller than the Margulis constant and surface S there exists a continuous map $\rho_0 : [0, \infty) \rightarrow [0, \infty)$ with $\rho_0(\epsilon) \leq \epsilon$ for every $\epsilon \geq 0$ where the following holds. Suppose Γ is an η -separated geodesic link in the hyperbolic 3-manifold N and $f : S \rightarrow N$ is a CAT(-1)-piecewise ruled map which is in addition 2-incompressible rel Γ . Then for every $\epsilon \leq \frac{\eta}{2}$*

$$f(\Sigma_f^{\leq \epsilon}) \subset N^{\leq \epsilon} \quad \text{and} \quad f(\Sigma_f^{\geq \epsilon}) \subset N^{\geq \rho_0(\epsilon)},$$

where Σ_f denotes the surface S equipped with the path metric induced from the map f .

Proof. The first inclusion is what we have already proved in lemma 3.4. Also note that our argument in the proof of the Bounded Diameter Lemma 3.5 shows that diameter of every component of $\Sigma_f^{\geq \epsilon}$ is at most $D = -2\pi\chi(S)/\epsilon$. Assume $\epsilon < \eta/8$; then lemma 3.7 shows the fundamental group of each component of $\Sigma_f^{\geq \epsilon}$ contains a free group of rank 2.

Choose $\delta \leq \epsilon$ to be small enough so that $\xi(\eta/\delta) \geq D$, where $\xi(\cdot)$ is the function obtained in lemma 2.1, and suppose the image of a point $x \in \Sigma_f^{\geq \epsilon}$ is contained in $N^{< \delta}$. By lemma 2.1, the image of the component R of $\Sigma_f^{\geq \epsilon}$ which contains x , is contained in a component T of $N^{\leq \eta}$. Similar to the proof of the above lemma, replace f by the induced π_1 -injective map $\bar{f} : \Sigma_f \rightarrow N \setminus \bar{\Gamma}$. Then $\bar{f}(R) \subset \bar{T}$ where \bar{T} is the pre-image of T . But $\pi_1(\bar{T})$ is abelian and since $\pi_1(R)$ contains a free group of rank 2, we have a contradiction to the fact that \bar{f} is π_1 -injective. Therefore $f(x) \in N^{\geq \rho_0(\epsilon)}$ where

$$\rho_0(\epsilon) = \min \left(\epsilon, \eta_1 / \xi^{-1} \left(\frac{-2\pi\chi(S)}{\epsilon} \right) \right).$$

When $\epsilon > \eta/8$ then obviously $\Sigma_f^{\geq \epsilon} \subset \Sigma_f^{\geq \eta/8}$ and we can simply define

$$\rho_0(\epsilon) = \rho_0(\eta/8).$$

□

4. QUASI-CONVEXITY

Suppose N is a hyperbolic 3-manifold and Γ is an η -separated geodesic link for some $\eta > 0$. Also assume S is the interior of a surface of finite type and $j : S \rightarrow N$ is a map that is 2-incompressible rel Γ .

Definition 4.1. Given $B > 0$, we define $\mathcal{C}(B, j) \subset \mathcal{C}_0(S)$ to be the set of all elements $\alpha \in \mathcal{C}_0(S)$ whose Γ -geodesic representative in N has length at most B .

Our main goal in this section is to show that for B larger than the Bers' constant, $\mathcal{C}(B, j)$ is quasi-convex with a quasi-convexity constant which depends only on $\chi(S), \eta$ and B .

Given a $\text{CAT}(-1)$ -piecewise ruled map $f : S \rightarrow N$ and $B > 0$, by $\mathbf{short}(f, B)$ we denote the set of elements in $\mathcal{C}_0(S)$ whose geodesic representative on the metric induced by f has length at most B . Using this, we define a projection $\Pi_{j,B}(\alpha)$ for every $\alpha \in \mathcal{C}_0(S)$ as follows:

$$\Pi_{j,B}(\alpha) = \bigcup_f \mathbf{short}(f, B),$$

where the union is over all $\text{CAT}(-1)$ -piecewise ruled maps f which are Γ -homotopic to j and realize α . Note that by Theorem 3.2, the set of such maps is nonempty.

Our first result is to show that $\mathbf{short}(f, B)$ has bounded diameter in $\mathcal{C}(S)$ for a single element f used in the above union.

Lemma 4.1. *Suppose $f : S \rightarrow N$ is a $\text{CAT}(-1)$ -piecewise ruled map which is 2-incompressible rel $\Gamma \subset N$, an η -separated geodesic link. For every $\alpha, \beta \in \mathbf{short}(f, B)$, $i(\alpha, \beta)$ is bounded by a constant that depends only on $\chi(S), \eta$ and B . In particular $\text{diam}_{\mathcal{C}}(\mathbf{short}(f, B))$ is bounded by a constant $d_{\chi(S), \eta, B}$.*

Proof. Assume Σ_f is the surface S equipped with the $\text{CAT}(-1)$ metric induced from the map f . We consider all the lengths and distances in this metric and assume α and β are geodesic closed curves. Consider $\epsilon > 0$ very small in a way that will be determined in what follows. Also suppose

$$i(\alpha, \beta) > \lceil B/\epsilon \rceil^2$$

where $\lceil x \rceil$ denotes the smallest integer that is greater than or equal to x . We can divide α into $\lceil B/\epsilon \rceil$ segments of length at most ϵ and by Pigeonhole Principle, there will be at least $\lceil B/\epsilon \rceil + 1$ points of $\alpha \cap \beta$ in one of these segments. Similarly we can divide β into $\lceil B/\epsilon \rceil$ segments of length at most ϵ and among the points of $\alpha \cap \beta$ which we have already chosen there will be two points p, q in one of these segments. This way we find p and q in $\alpha \cap \beta$ with an arc $\kappa_\alpha \subset \alpha$ and an arc $\kappa_\beta \subset \beta$ both with lengths at most ϵ and endpoints p and q . The concatenation of these arcs gives a closed curve $\gamma = \kappa_1 * \kappa_2$ of length at most 2ϵ . This is an essential closed curve on S and by lemma 3.4, its image by f must also be essential in N , provided that $\epsilon \leq \eta/2$. Even more, the image of this loop is contained in $N^{\leq \epsilon}$ and so does $f(p)$.

Let $\delta = \rho_0(\eta/2)$ where $\rho_0(\cdot)$ is the function obtained in Proposition 3.8 and suppose ϵ is small enough such that

$$\xi(\delta/\epsilon) > B$$

where $\xi(\cdot)$ is the lower bound for the distance between boundaries of Margulis tubes in lemma 2.1. Since $f(p) \in N^{\leq \epsilon}$ and the lengths of $f(\alpha)$ and $f(\beta)$ is at most B , we have $f(\alpha), f(\beta) \subset N^{\leq \delta}$. Then because of Proposition 3.8 and since $\delta = \rho_0(\eta/2)$, we must have $\alpha, \beta \subset \Sigma_f^{\leq \eta/2}$. They intersect; so they have to belong to the same component of $\Sigma_f^{\leq \eta/2}$. However by lemma 3.7, such a component is an essential annulus. But α and β are simple; so they have to be freely homotopic to the core curve of the annulus and $i(\alpha, \beta) = 0$. This contradicts our assumption and we must have $i(\alpha, \beta) \leq \lceil B/\epsilon \rceil^2$.

□

Proposition 4.2. *Suppose $N, \Gamma, j : S \rightarrow N$ are as above and $B_0 > 0$ is the Bers' constant for S . There exists a constant c depending only on $\chi(S)$ and η such that*

- (Coarse idempotence) *If $\alpha \in \mathcal{C}(B_0, j)$ then $\alpha \in \Pi_{j, B_0}(\alpha)$.*
- (Coarse Lipschitz) *For α and $\beta \in \mathcal{C}_0(S)$ with $d_{\mathcal{C}}(\alpha, \beta) \leq 1$,*

$$\text{diam}_{\mathcal{C}}(\Pi_{j, B_0}(\alpha) \cup \Pi_{j, B_0}(\beta)) \leq c.$$

Proof. The proof of (a) follows immediately from Theorem 3.2.

For part (b), note that by applying Theorem 3.2 again, one knows there exists a $\text{CAT}(-1)$ -piecewise ruled map which is Γ -homotopic to j and realizes $\alpha \cup \beta$. Also by lemma 4.1, we know that the diameter of $\mathbf{short}(f, B_0)$ is bounded for every $\text{CAT}(-1)$ -piecewise ruled map f which is 2-incompressible rel Γ . So it is enough to show that if f and g both realize α (and similarly if they both realize β), the distance between $\mathbf{short}(f, B_0)$ and $\mathbf{short}(g, B_0)$ is bounded by a constant depending on $\chi(S)$ and η . In lemma 4.3 which follows, we prove that in this situation, there exists a non-contractible simple loop γ whose length both in Σ_f and Σ_g is bounded by a function of $\chi(S)$ and η . Then by lemma 4.1, the distances in $\mathcal{C}(S)$ between γ and $\mathbf{short}(f, B_0)$ and also between γ and $\mathbf{short}(g, B_0)$ are bounded depending only on $\chi(S)$ and η . This finishes the proof. \square

Lemma 4.3. *Suppose $j : S \rightarrow N$ is 2-incompressible rel Γ , an η -separated geodesic link in the hyperbolic 3-manifold N , and $\eta > 0$ is smaller than the Margulis constant. If $f, g : S \rightarrow N$ are both $\text{CAT}(-1)$ -piecewise ruled maps, both are Γ -homotopic to j and both realize a non-contractible simple loop α , there exists a non-contractible simple loop γ whose length in Σ_f and Σ_g is bounded by a constant that depends only on $\chi(S)$ and η .*

Proof. By Proposition 3.8,

$$f(\Sigma_f^{\geq \epsilon}) \subset N^{\geq \rho_0(\epsilon)}$$

for every $\epsilon \in (0, \eta/2]$. Assume α represents the geodesic representative of α in Σ_f and Σ_g . Since f and g take these geodesic representatives to the unique Γ -geodesic representative of α in N , we use f and g and identify copies of α on Σ_f and Σ_g . Also similar to our previous arguments, realize the maps f and g as maps $\bar{f} : \Sigma_f \rightarrow \overline{N \setminus \Gamma}$ and $\bar{g} : \Sigma_g \rightarrow \overline{N \setminus \Gamma}$ respectively.

Assume α enters the δ -thin part of Σ_f where $\delta = \rho_0(\eta/8)$ and assume p is a point of $\alpha \cap \Sigma_f^{\leq \delta}$. Then by Proposition 3.8, $f(p) \in N^{\leq \delta}$ and since $f(p) = g(p)$, using Proposition 3.8 once more, we have $p \in \Sigma_g^{\leq \eta/8}$. As a result, there exist non-contractible simple loops γ_f, γ_g based at p where the length of γ_f in Σ_f and the length of γ_g in Σ_g do not exceed $\eta/8$. Let T be the component of $\overline{N \setminus \Gamma}^{\leq \eta/8}$ which contains $f(p) = \bar{g}(p)$. We obviously have $\bar{f}(\gamma_f), \bar{g}(\gamma_g) \subset T$ and since \bar{f} and \bar{g} are homotopic, the loop $\bar{f}(\gamma_g)$ is also homotopic (as a loop based at $\bar{f}(p)$) into T and therefore $\bar{f}(\gamma_f)$ and $\bar{f}(\gamma_g)$ represent two elements of $\pi_1(T)$. Using lemma 3.6, we know $\pi_1(T)$ is abelian; so $\bar{f}(\gamma_f)$ and $\bar{f}(\gamma_g)$ commute. Since \bar{f} is incompressible, γ_f and γ_g must also commute; but they are simple and therefore they have to be

homotopic to each other. This provides a simple loop whose length in Σ_f and Σ_g is at most $\eta/2$ and we have proved the conclusion in this case.

Hence we can assume α does not enter $\Sigma_{\bar{f}}^{\leq \delta}$ and using a similar argument we also assume it does not enter $\Sigma_{\bar{g}}^{\leq \delta}$. Suppose $\kappa : [0, \ell] \rightarrow \alpha$ parametrizes α with respect to the arc length, where ℓ is the length of α and as mentioned before we are viewing α as a closed geodesic in Σ_f, Σ_g and N simultaneously.

Let $\nu = \rho_0(\delta)/4$ and $K = 1 + 4[\chi(S)]^2/\nu^4$; we claim that if $\ell \geq K$ then there exist $0 \leq a < b \leq K$ such that $\kappa(a)$ and $\kappa(b)$ are closer than 2ν both in Σ_f and Σ_g but subarcs of α connecting them have lengths ≥ 1 . To prove the claim, let $M = \Sigma_f \times \Sigma_g$ be the product 4-manifold with the product metric. The curvature of Σ_f and Σ_g at all points is ≤ 1 and therefore by Gauss-Bonnet, the area of each of them is at most $-2\pi\chi(S)$. By Fubini's Theorem, the volume of M is the product of these areas and is at most $4\pi^2[\chi(S)]^2$. Because $\alpha \subset \Sigma_{\bar{f}}^{\geq \delta}$ and $\alpha \subset \Sigma_{\bar{g}}^{\geq \delta}$ and $\nu < \delta$, for every point $(\kappa(t), \kappa(t)) \in M$ the product

$$D_M^\nu(\kappa(t), \kappa(t)) = B_{\Sigma_f}(\kappa(t), \nu) \times B_{\Sigma_g}(\kappa(t), \nu)$$

is an embedded product of disks in M , where $B_\Sigma(x, r)$ denotes the open disk of radius r centered at x in Σ . Again using the upperbound for the curvature, the areas of $B_{\Sigma_f}(\kappa(t), \nu)$ and $B_{\Sigma_g}(\kappa(t), \nu)$ are at least $\pi\nu^2$ and therefore the volume of $D_M^\nu(\kappa(t), \kappa(t))$ is at least $\pi^2\nu^4$.

If $\ell \geq K$ then, because of the upper bound for the volume of M , the sets

$$D_M^\nu(\kappa(n), \kappa(n)), \quad n \in \mathbb{Z} \cap [0, K-1]$$

cannot be all disjoint and therefore we find $0 \leq a < b \leq K$ with

$$D_M^\nu(\kappa(a), \kappa(a)) \cap D_M^\nu(\kappa(b), \kappa(b)) \neq \emptyset.$$

and such that the subarcs of α connecting $\kappa(a)$ and $\kappa(b)$ have length ≥ 1 . This immediately implies that the distance between $\kappa(a)$ and $\kappa(b)$ in both Σ_f and Σ_g strictly less than $2\nu = \rho_0(\delta)/2$. Choose an arc $\mu_f \subset \Sigma_f$ and an arc $\mu_g \subset \Sigma_g$ of length $< 2\nu$ with endpoints $\kappa(a)$ and $\kappa(b)$. By concatenating these arcs with $\kappa[a, b]$, we obtain loops

$$\gamma_f = \kappa[a, b] * \mu_f \subset \Sigma_f \quad \text{and} \quad \gamma_g = \kappa[a, b] * \mu_g$$

whose lengths are bounded by $K + 2\nu$. We claim these two loops represent freely homotopic loops in S . Because \bar{f} and \bar{g} are incompressible, it will be enough to show $\bar{f}(\mu_f)$ and $\bar{g}(\mu_g)$ are homotopic (rel endpoints) in $\overline{N \setminus \Gamma}$ (choose a base point on α).

The closed curve $\omega = f(\mu_f) * g(\mu_g)$ in N and its lift $\bar{\omega} = \bar{f}(\mu_f) * \bar{g}(\mu_g)$ in $\overline{N \setminus \Gamma}$ have lengths strictly less than $4\nu = \rho_0(\delta)$. We know $\kappa(a) \in \Sigma_{\bar{f}}^{\geq \delta}$ and by proposition 3.8, $f(\kappa(a)) \in N^{\geq \rho_0(\delta)}$. Hence every loop of length $< \rho_0(\delta)$ based at $f(\kappa(a))$ has to be homotopically trivial in N . By lemma 3.6, either $\bar{\omega}$ is homotopically trivial in $\overline{N \setminus \Gamma}$ or ω is a multiplied meridian of a component of Γ in N . However, we know \bar{f} and \bar{g} are homotopic maps in $\overline{N \setminus \Gamma}$. Even more an easy corollary of our proof and construction in Theorem 3.2 is that we can assume the intermediate maps in the homotopy between \bar{f} and \bar{g} all realize α , i.e. they send α to its Γ -geodesic

representative in $\overline{N \setminus \Gamma}$. In particular, the homotopy restricted to μ_g provides a map $\bar{H} : [0, 1] \times [0, 1] \rightarrow \overline{N \setminus \Gamma}$ where the image of $[0, 1] \times \{0\}$ is the arc $\bar{g}(\mu_g)$, the image of $\{0, 1\} \times [0, 1]$ is a subset of $\bar{f}(\alpha) = \bar{g}(\alpha)$ and finally the image of $[0, 1] \times \{1\}$ is a subset of $\bar{f}(\Sigma_f)$. In particular the arc $\bar{g}(\mu_g)$ is homotopic (rel endpoints) to the arc

$$\bar{H}(\{0\} \times [0, 1]) * \bar{H}([0, 1] \times \{1\}) * \bar{H}(\{1\} \times [0, 1]) \subset \bar{f}(\Sigma_f)$$

in $\overline{N \setminus \Gamma}$.

Let $H : [0, 1] \times [0, 1] \rightarrow N$ denotes the composition of \bar{H} with the projection $\overline{N \setminus \Gamma} \rightarrow N$. Then the closed curve

$$\mu_f * H(\{0\} \times [0, 1]) * H([0, 1] \times \{1\}) * H(\{1\} \times [0, 1])$$

is either homotopically trivial or is a multiplied meridian of a component of Γ . But this loop is in $g(\Sigma_g)$ and since g is 2-incompressible it must be homotopically trivial in $g(\Sigma_g)$. This proves $\bar{f}(\mu_f)$ and $\bar{g}(\mu_g)$ are homotopic (rel endpoints) and as a result, γ_f and γ_g are freely homotopic in S . So they represent a free homotopy class whose length in Σ_f and Σ_g is bounded by $K + 2\nu$, which depends only on $\chi(S)$ and η . It is possible that this homotopy class does not represent a simple loop in S but if we put it in minimal position and use surgery, we obtain at least one simple loop whose length in Σ_f and Σ_g is smaller than this non-simple loop.

The only remaining case is when $\ell < K$, i.e. length of α in both Σ_f and Σ_g is less than K . But this again provides a curve (α itself) whose length both in Σ_f and Σ_g is bounded from above by a function of $\chi(S)$ and η .

□

Note that we have proved the proposition 4.2 for the constant B_0 . However using lemma 4.1, one can easily extend it for every $B \geq B_0$.

Corollary 4.4. *Suppose $j : S \rightarrow N$ is 2-incompressible rel Γ , an η -separated geodesic link in the hyperbolic 3-manifold N and $B \geq B_0$ is given. There exists a constant c_B depending only on $\chi(S), \eta$ and B such that*

- (Coarse idempotence) *If $\alpha \in \mathcal{C}(B, j)$ then $\alpha \in \Pi_{j,B}(\alpha)$.*
- (Coarse Lipschitz) *For α and $\beta \in \mathcal{C}_0(S)$ with $d_{\mathcal{C}}(\alpha, \beta) \leq 1$,*

$$\text{diam}_{\mathcal{C}}(\Pi_{j,B}(\alpha) \cup \Pi_{j,B}(\beta)) \leq c_B.$$

Recall that $\mathcal{C}(B, j)$ denotes the set of simple closed curves on S whose Γ -geodesic representatives have length $\leq B$. Once we have the above properties of the projection, we can use the following observation of Minsky [Min01, Lem 3.3].

Lemma 4.5. *Let X be a δ -hyperbolic geodesic metric space and $Y \subset X$ a subset admitting a map $\Pi : X \rightarrow Y$ which is coarse-Lipschitz and coarse idempotent. That is, there exists $C > 0$ such that*

- *if $d(x, y) \leq 1$ then $d(\Pi(x), \Pi(y)) \leq C$, and*
- *if $y \in Y$ then $d(y, \Pi(y)) \leq C$.*

Then Y is K -quasi-convex for some $K = K(\delta, C)$, and furthermore if g is a geodesic in X whose endpoints are within distance a of Y then

$$d(x, \Pi(x)) \leq b$$

for some $b = b(a, \delta, C)$.

We use this to prove the quasi-convexity of $\mathcal{C}(B, j)$ which obviously implies Theorem 1.1.

Theorem 4.6 (Quasi-convexity Theorem). *Suppose $j : S \rightarrow N$ is 2-incompressible rel Γ , an η -separated geodesic link in N . Given $B \geq B_0$, there exists L depending only on $\chi(S), \eta$ and B such that the set $\mathcal{C}(B, j)$ is L -quasi-convex in $\mathcal{C}(S)$. Moreover if β is a geodesic in $\mathcal{C}(S)$ with endpoints in $\mathcal{C}(B, j)$ then $d_{\mathcal{C}}(x, \Pi_{j,B}(x)) \leq L$ for each $x \in \beta$.*

5. FINDING GEODESIC LINKS

Like in the previous section, let S be the interior of an oriented surface of finite type and with negative Euler characteristic. Also let P be a hyperbolic 3-manifold homeomorphic to $S \times \mathbb{R}$. Recall that P is *doubly degenerate* if it is identical to its convex core. It follows from work of Thurston and Canary that in this situation, one can *sweep out* through P by simplicial hyperbolic surfaces, i.e. there exists a continuous map $S \times \mathbb{R} \rightarrow P$ which is onto, induces a homotopy equivalence and the image of $S \times \{t\}$ is a simplicial hyperbolic surface for every $t \in \mathbb{R}$. The large scale geometry of such 3-manifolds is understood by works of Brock-Canary-Minsky [ELC1, ELC2] and in particular there are bi-Lipschitz models of these 3-manifolds which are constructed by using the combinatorial information provided from the complex of curves and works of Masur-Minsky [MM99, MM00].

Suppose N is a hyperbolic 3-manifold and the surface S is fixed; by an *L -product region* in N , we mean a submanifold $U \subset N$ homeomorphic to $S \times [-1, 1]$ such that there is a doubly degenerate hyperbolic 3-manifold P homeomorphic to $S \times \mathbb{R}$ and a homotopy equivalence $\phi : U \rightarrow P$ with the following properties:

- ϕ is a L -bi-Lipschitz embedding, and
- there are simplicial hyperbolic surfaces $X_-, X_+ \subset P$ such that $\phi(S \times \{\pm 1\}) \subset \mathcal{N}(X_{\pm})$.

The ϵ -width of U is the minimal length, relative to $N^{\leq \epsilon}$, of an arc in U joining the two boundary components and avoiding the cusps. A *level surface* $\Sigma \subset U$ is an embedded surface such that the inclusion $\Sigma \hookrightarrow U$ is a homotopy equivalence.

The goal of this section is to show that given an embedding of a surface into a hyperbolic 3-manifold, large product regions can be used to find η -separated geodesic links which make the embedding 2-incompressible. Even more the total length of the obtained geodesic link will be bounded depending on $\chi(S)$ and η will be bounded below by a universal constant. First we need to show that in a wide product region we can find lots of closed geodesic curves with definite tube radii and large pairwise distances.

Using the Bounded Diameter Lemma 3.5, we know there is a bound $D(\chi(S), \epsilon)$ for the diameter of every simplicial hyperbolic surface relative to the ϵ -thin parts. Now we claim the following proposition. Recall that given S there exists the Bers' constant B_0 depending only on $\chi(S)$ and such that for every CAT(-1) metric on S there exists a pants decomposition whose total length is bounded from above by B_0 .

Proposition 5.1. *Given n, L and ϵ_0 if ω is sufficiently large, in an L -product region $U \subset N$ of ϵ_0 -width ω there exists a collection $\gamma_1, \gamma_2, \dots, \gamma_n$ of closed geodesics and an embedding $\psi : S \times [1, n+1] \rightarrow U$ such that*

- ψ induces a homotopy equivalence,
- each γ_i has tube radius ≥ 0.025 and length $\leq LB_0$, $i = 1, \dots, n$,
- $\gamma_i \subset \psi(S \times (i, i+1))$, $i = 1, \dots, n$,
- the diameter, rel $N^{\leq \epsilon_0}$, of each surface $\psi(S \times \{i\})$ is bounded by a constant $C(\chi(S), \epsilon_0, L)$ depending only on L, ϵ_0 and $\chi(S)$,
- the distance rel $N^{\leq \epsilon_0}$ between $\psi(S \times \{i\})$ and $\psi(S \times \{i+1\})$ is at least $D(\chi(S), \epsilon_0)$, $i = 1, \dots, n$, where $D(\chi(S), \epsilon_0)$ is the constant in lemma 3.5 and
- the distance between γ_i and each of the surfaces $\psi(S \times \{i\})$ and $\psi(S \times \{i+1\})$ is at least 1, $i = 1, \dots, n$.

Proof. By definition there is a doubly degenerate hyperbolic 3-manifold P and a homotopy equivalence $\phi : U \rightarrow P$ which is an L -bi-Lipschitz embedding. Let $Q = \phi(U)$; then obviously Q is a 1-product region and it makes sense to speak of the ϵ -width of Q for any $\epsilon > 0$. When ϵ_0 -width of U is at least ω then the ϵ_1 -width of Q is at least ω/L for $\epsilon_1 = \epsilon_0/L$. We know that every point of P belongs to a simplicial hyperbolic surface homeomorphic to S whose diameter rel $P^{\leq \epsilon}$ is at most $D(\chi(S), \epsilon)$ for any $\epsilon > 0$. Also on every such surface there exists an essential simple closed curve of length $\leq B_0$. Even more using a theorem of Freedman-Hass-Scott [FHS], we know that in every neighborhood of one of these simplicial hyperbolic surfaces, we can find an embedded level surface. Also by a theorem of Waldhausen, if two such embedded surfaces are disjoint then they enclose a submanifold of Q homeomorphic to $S \times [0, 1]$.

Using the above information and for a given $\omega_1 > 0$ and if

$$\omega/L \geq (2n+5)\omega_1 D(\chi(S), \epsilon_1),$$

we can find n essential closed curves $\bar{\alpha}_1, \dots, \bar{\alpha}_n$ in Q and an embedding $\bar{\psi} : S \times [1, n+1] \rightarrow Q$ such that

- $\bar{\psi}$ induces a homotopy equivalence,
- each $\bar{\alpha}_i$ has length $\leq B_0$ and is homotopic to a simple closed curve on S ,
- $\bar{\alpha}_i \subset \bar{\psi}(S \times (i, i+1))$, $i = 1, \dots, n$,
- the diameter, rel $P^{\leq \epsilon_1}$, of each surface $\bar{\psi}(S \times \{i\})$ is bounded by $D(\chi(S), \epsilon_1)$, $i = 1, \dots, n+1$,
- the distance, rel $P^{\leq \epsilon_1}$, between $\bar{\psi}(S \times \{i\})$ and $\bar{\psi}(S \times \{i+1\})$ is at least ω_1 , $i = 1, \dots, n$ and

- the distance, rel $P^{\leq \epsilon_1}$, between $\bar{\alpha}_i$ and the surfaces $\bar{\psi}(S \times \{i\})$ and $\bar{\psi}(S \times \{i+1\})$ is at least ω_1 , $i = 1, \dots, n$.

Now we consider the map $\psi = \phi^{-1} \circ \bar{\psi} : S \times [1, n+1] \rightarrow U$ and we have closed curves $\alpha_1 = \phi(\bar{\alpha}_1), \alpha_2 = \phi(\bar{\alpha}_2), \dots, \alpha_n = \phi(\bar{\alpha}_n)$ such that for $\epsilon_2 = \epsilon_1/L$ we have

- ψ induces a homotopy equivalence,
- length of each α_i is at most B_0L ,
- $\alpha_i \subset \psi(S \times (i, i+1)), i = 1, \dots, n$,
- the diameter, rel $U^{\leq \epsilon_2}$, of each surface $\psi(S \times \{i\})$ is bounded by $LD(\chi(S), \epsilon_1)$, $i = 1, \dots, n+1$,
- the distance, rel $U^{\leq \epsilon_2}$, between $\psi(S \times \{i\})$ and $\psi(S \times \{i+1\})$ is at least ω_1/L and
- the distance, rel $U^{\leq \epsilon_2}$, between α_i and the surfaces $\psi(S \times \{i\})$ and $\psi(S \times \{i+1\})$ is at least ω_1/L , $i = 1, \dots, n$.

One bothering feature of the above collection is that the lower bounds are obtained for distances rel $N^{\leq \epsilon_2}$ rather than for distances rel $N^{\leq \epsilon_0}$. This is however fixable using the next lemma:

Lemma 5.2. *Given $0 < \epsilon_1, \epsilon_2$ both smaller than the Margulis constant, there exists a monotone map $G_{\epsilon_1, \epsilon_2} : [0, \infty) \rightarrow [0, \infty)$ with $G_{\epsilon_1, \epsilon_2}(0) = 0$ and $\lim_{t \rightarrow \infty} G_{\epsilon_1, \epsilon_2}(t) = \infty$ such that for every hyperbolic 3-manifold N and $x, y \in N$*

$$d_N^{\leq \epsilon_1}(x, y) \leq G_{\epsilon_1, \epsilon_2}(d_N^{\leq \epsilon_2}(x, y)).$$

Proof. If $\epsilon_1 \geq \epsilon_2$ then obviously

$$d_N^{\leq \epsilon_1}(x, y) \leq d_N^{\leq \epsilon_2}(x, y)$$

and we can simply define $G_{\epsilon_1, \epsilon_2}$ to be the identity map when $\epsilon_1 \geq \epsilon_2$. So assume $\epsilon_1 < \epsilon_2$.

By lemma 2.2 one can see that if T_2 is a component of $N^{\leq \epsilon_2}$ which contains a component T_1 of $N^{\leq \epsilon_1}$ then the distance from every point of T_2 to T_1 is bounded by a function of ϵ_1 and ϵ_2 . In particular if the length of the core geodesic of a Margulis tube T in $N^{\leq \epsilon_2}$ is bigger than or equal to ϵ_1 then the diameter of T is bounded by a function of ϵ_1 and ϵ_2 .

Let g be a path connecting x and y which provides their distance rel $N^{\leq \epsilon_2}$, i.e.

$$(5.1) \quad d^{\leq \epsilon_2}(x, y) = l(g) - l(g \cap N^{\leq \epsilon_2}),$$

and $l(g \cap N^{\leq \epsilon_2})$ denotes the sum of the lengths of the subsegments of g inside components of $N^{\leq \epsilon_2}$. Assume T_2 is a component of $N^{\leq \epsilon_2}$ with $g \cap T_2 \neq \emptyset$ and $x', y' \in \partial T_2 \cap g$ are the entry and exit points of g . Obviously we are allowed to replace $g[x', y']$, the segment of g connecting x' and y' , with any other path inside T_2 that connects them. When the core curve of T_2 has length $\geq \epsilon_1$, i.e. $T \cap N^{\leq \epsilon_1} = \emptyset$, we assume $g[x', y']$ is the shortest geodesic connecting x' and y' . Then it stays in T_2 (by convexity of T_2) and its length is bounded by the diameter of T_2 which is bounded by a function of ϵ_1 and ϵ_2 by our previous remark. If the core curve of T_2 has length $< \epsilon_1$, i.e. $T \cap N^{\leq \epsilon_1} \neq \emptyset$, let $T_1 \subset T_2$ be the associated component of $N^{\leq \epsilon_1}$. In this case, we assume $g[x', y']$ is the concatenation of three segments

$\alpha * \beta * \gamma$ where α is the shortest path connecting x' to ∂T_1 , γ is the shortest path connecting y' to ∂T_1 and β is a path in T_1 that connects the endpoints of α and γ on ∂T_1 . Again by our earlier remark, lengths of α and γ are bounded by a function of ϵ_1 and ϵ_2 and as a result of all these

$$(5.2) \quad l(g \cap N^{\leq \epsilon_2}) - l(g \cap N^{\leq \epsilon_1}) \leq C(\epsilon_1, \epsilon_2) |g \cap N^{\leq \epsilon_2}|$$

where $C(\epsilon_1, \epsilon_2)$ is a constant depending only on ϵ_1 and ϵ_2 and the last quantity is the number of components of $N^{\leq \epsilon_1}$ which intersect g .

However by our assumption about the Margulis constant, the distance between every two components of $N^{\leq \epsilon_2}$ is at least 1 and therefore

$$(5.3) \quad d^{\leq \epsilon_2}(x, y) = l(g \cap N^{\leq \epsilon_2}) \geq |g \cap N^{\leq \epsilon_2}| - 1.$$

Now using (5.1), (5.2) and (5.3) we have

$$\begin{aligned} d_N^{\leq \epsilon_1}(x, y) &\leq l(g) - l(g \cap N^{\leq \epsilon_1}) \\ &\leq l(g) - l(g \cap N^{\leq \epsilon_2}) + C(\epsilon_1, \epsilon_2) |g \cap N^{\leq \epsilon_1}| \\ &\leq d^{\leq \epsilon_2}(x, y) + C(\epsilon_1, \epsilon_2) (d^{\leq \epsilon_2}(x, y) + 1) \\ &= (C(\epsilon_1, \epsilon_2) + 1) d^{\leq \epsilon_2}(x, y) + C(\epsilon_1, \epsilon_2) \end{aligned}$$

which finishes the proof of the lemma. \square

Using this lemma we can replace the lower bound ω_1/L for the distances rel N^{ϵ_2} with the lower bound $\omega_2 = G_{\epsilon_0, \epsilon_2}(\omega_1/L)$ for the distances rel $N^{\leq \epsilon_0}$. Since $\epsilon_2 = \epsilon_0/L^2$ this depends only on our choice of ω_1, ϵ_0 and L . Besides it tends to infinity as $\omega_1 \rightarrow \infty$.

By assuming that ω is large, we can make sure that the above collection of closed curves is constructed in a way that its distance to ∂U is at least B_0L . Then it follows from lemma 3.3 that each $\alpha_i, i = 1, \dots, n$, is homotopically nontrivial in N . Even more the geodesic representatives of $\alpha_1, \dots, \alpha_n$ in N , which we respectively denote by $\alpha_1^*, \dots, \alpha_n^*$, are within U and by lemma 2.2

$$d_N^{\leq \epsilon_0}(\alpha_i, \alpha_i^*) \leq c_1 = \cosh^{-1}(B_0L/\epsilon_0), \quad i = 1, \dots, n.$$

The length of each of the curves α_i^* is still bounded from above by B_0L . Next we want to change each curve α_i^* to guarantee that the tube radius(α_i^*) ≥ 0.025 for every i . Meyerhoff [Me87] defines a monotonically decreasing function $r : (0, 0.1] \rightarrow [0.3, \infty)$ such that if γ is a closed geodesic in N and $l_N(\gamma) \leq t$, the tube radius(γ) $\geq r(t)$. So if $l_N(\alpha_i^*) \leq 0.1$ then the tube radius(α_i^*) ≥ 0.3 and we are satisfied with its tube radius.

Following Gabai [Ga97] assume $l_N(\alpha_i^*) \geq 0.1$ and tube radius(α_i^*) $\leq 0.025 \leq \frac{1}{4}l_N(\alpha_i^*)$ then there exists a geodesic β_i^* homotopic to a curve which is a union of a segment of α_i^* and an orthogonal arc from α_i^* to itself, and each of these segments has length $\leq l_N(\alpha_i^*)/2$. By straightening these segments and using the law of cosines, we see that since $l(\alpha_i^*) \geq 0.1$, $l(\beta_i^*) < l(\alpha_i^*) - 0.02$. Note that

$$d_N^{\leq \epsilon_0}(\alpha_i^*, \beta_i^*) \leq c_1 + 0.025$$

by another application of lemma 2.2. If the tube radius $(\beta_i) \leq .025$ then we can repeat the same procedure and obtain a shorter closed geodesic and repeat the same procedure if necessary. However such a process reduces the length by at least 0.02 and since the length of the original curve was bounded from above by B_0L the process must stop after at most $B_0L/0.02$ times. At the end of the process we will be left with a closed geodesic γ_i with tube radius ≥ 0.025 .

As we have seen before, the distance $\text{rel } N^{\leq \epsilon_0}$ between two closed geodesics that appear consecutively is at most $c_1 + 0.025$ and the length of each of them is bounded from above by B_0L and therefore

$$d_N^{\leq \epsilon_0}(\alpha_i, \gamma_i) \leq c_2 = \left(\frac{B_0L}{0.02} + 1\right)(c_1 + 0.025 + B_0L)$$

for every $1 \leq i \leq n$. So provided that

$$\omega_2 > c_2 + B_0L + D(\chi(S), \epsilon_0) + 1$$

for every $i = 1, \dots, n$, the geodesic γ_i will stay in $\psi(S \times (i, i+1))$. Also the distance, $\text{rel } N^{\leq \epsilon_0}$, between $\psi(S \times \{i\})$ and $\psi(S \times \{i+1\})$ is at least $D(\chi(S), \epsilon_0)$ and the distance between γ_i and the surfaces $\psi(S \times \{i\})$ and $\psi(S \times \{i+1\})$ is at least 1. We already know ψ induces a homotopy equivalence between $S \times [1, n+1]$ and U and the diameter, $\text{rel } N^{\leq \epsilon_0}$, of each surface $\psi(S \times \{i\})$ is at most $G_{\epsilon_0, \epsilon_2}(LD(\chi(S), \epsilon_1))$. Hence proposition 5.1 follows. \square

Once we know the proposition 5.1, we turn our attention to proving the main result of this section and as a corollary prove Theorem 1.2. Given an embedding $j : S \rightarrow N$ into the oriented hyperbolic 3-manifold N , recall from the introduction that $j(S)$ is two-sided in N and using the orientations of S and N one can identify them as positive and negative sides of $j(S)$ in N . We say an embedding $g : (D^2, \partial D^2) \rightarrow (n, j(S))$ is a *compression to the positive side* (resp. *negative side*) of $j(S)$ if the image of ∂D^2 is a non-contractible loop on $j(S)$, the g -image of the interior of D^2 does not intersect $j(S)$, $g(D^2)$ is transverse to $j(S)$ and stays on the positive (resp. negative) side of $j(S)$. We say $j(S)$ or the embedding $j : S \rightarrow N$ is *incompressible to the positive side* (resp. *to the negative side*) when there is no compression to the positive side (resp. negative side) of $j(S)$. By an application of the Loop Theorem, such an embedding is incompressible if and only if it is incompressible both to the positive and negative sides.

Also recall that $j(S)$ is blocked from the positive (resp. negative) side by a product region U , called the *block*, if $j(S)$ does not intersect U and it is homotopic to a component of ∂U with a homotopy that stays on the positive (resp. negative) side of $j(S)$. When the block is an L -product region and ϵ_0 is chosen, by *width* of the block we mean the ϵ_0 -width of the product region. As in the introduction, we develop the convention to say $j(S)$ is blocked from the positive (resp. negative) by the empty set if it is incompressible to the positive (resp. negative) side of $j(S)$ and also the width of the empty set is infinite. The following theorem is a more precise and general restatement of Theorem 1.2.

Theorem 5.3. *Suppose $\epsilon_0, L > 0$ are fixed. Suppose $j : S \rightarrow N$ is a proper embedding of the oriented surface S with $\chi(S) < 0$ into an oriented hyperbolic 3-manifold N . Also assume $j(S)$ is blocked by a block U^+ from the positive side and*

by a block U^- from the negative side. There exists ω depending on $\chi(S), \epsilon_0$ and L such that if U^+ and U^- are L -product regions of ϵ_0 -width ω then there exists a 0.025-separated geodesic link $\Gamma \subset (U^+ \cup U^-)$ such that j is 2-incompressible rel Γ . Even more the length of each component of Γ is bounded depending on $\chi(S), L$ and the number of its components depends only on $\chi(S)$.

Proof. We assume j is not incompressible to either positive or negative sides. The argument for the other cases follows in a similar manner.

Suppose $n = 3|\chi(S)| + 1$ and as in the hypothesis, assume U^+ and U^- are L -product regions of ϵ_0 -width ω which block $j(S)$ from the positive and negative sides. By proposition 5.1, if ω is sufficiently large we have a geodesic link $\Gamma^+ = \{\gamma_1^+, \dots, \gamma_n^+\}$ in U^+ and $\Gamma^- = \{\gamma_1^-, \dots, \gamma_n^-\}$ in U^- and also embeddings $\psi^\pm : S \rightarrow [1, n+1] \rightarrow U^\pm$ which satisfy the conclusion of the proposition 5.1.

If we remove the interiors of $\psi^+(S \times [1, n+1])$ and $\psi^-(S \times [1, n+1])$ then we are left with a compact submanifold of N which contains $j(S)$. Furthermore the two boundary components of this submanifold are homotopic to $j(S)$ and therefore to each other within the submanifold. As a consequence and by Waldhausen's Theorem, one knows that this submanifold is homeomorphic to $S \times [-1, 1]$. Using this homeomorphism, the embeddings ψ^+ and ψ^- and after possibly re-enumerating elements of Γ^+ and Γ^- , we have a submanifold of N identified with $S \times [-n-1, n+1]$ such that

- $S \times \{0\} = j(S)$,
- $\gamma_i^+ \subset S \times (i, i+1)$ and $\gamma_i^- \subset S \times (-i-1, -i)$, $i = 1, \dots, n$,
- the diameter, rel $N^{\leq \epsilon_0}$, of each $S \times \{i\}$ is bounded by $C(\chi(S), \epsilon_0, L)$, $i = \pm 1, \dots, \pm(n+1)$,
- the distance, rel $N^{\leq \epsilon_0}$, between $S \times \{i\}$ and $S \times \{i+1\}$ is at least $D(\chi(S), \epsilon_0)$, $i = -n-1, \dots, -2$ or $i = 1, \dots, n$ and
- the distances of γ_i to the surfaces $S \times \{i\}$ and $S \times \{i+1\}$ are at least 1, $i = -n-1, \dots, -2$ or $i = 1, \dots, n$.

We claim j is 2-incompressible rel $\Gamma = \Gamma^+ \cup \Gamma^-$. This proves Theorem 5.3 in this case, because tube radii of elements of Γ are bounded from below by 0.025, the distance between every two of them is bounded from below and the length of each of them is bounded from above by LB_0 . So Γ is 0.025-separated and obviously the length and number of components of Γ are bounded depending on $\chi(S)$ and L .

We say a surface $R \subset N$ separates γ_i^+ from U^- , $i = 1, \dots, n$, if for every arc κ in $S \times [-n-1, n+1]$ that connects γ_i^+ and U^- and $\partial\kappa \cap R = \emptyset$, $\langle R, \kappa \rangle \neq 0$ where $\langle \cdot, \cdot \rangle$ denotes the algebraic intersection number. Obviously $j(S) = S \times \{0\}$ separates every γ_i^+ from U^- , $i = 1, \dots, n$. If j is not 2-incompressible rel Γ , we can compress it via a compression which either misses Γ or crosses Γ once at $\alpha_1 \in \Gamma$. If possible, compress again via a compression meeting $\Gamma \setminus \alpha_1$ at most once, say at $\alpha_2 \in \Gamma$. After at most $|\chi(S)|$ such compressions and deleting 2-spheres we are left with embedded connected surfaces S_1, \dots, S_r ($r \leq |\chi(S)|$), none of which is a 2-sphere and each is 2-incompressible rel $\Gamma_1 = \Gamma \setminus \{\alpha_1, \dots, \alpha_s\}$ ($s \leq |\chi(S)|$). Each $\gamma \in \Gamma^+ \cap \Gamma_1$ is separated from U^- by exactly one of the surfaces S_1, \dots, S_r . Hence

there is one of them, say S_1 , which separates at least three distinct closed geodesics $\gamma_i^+, \gamma_j^+, \gamma_k^+ \in \Gamma^+ \cap \Gamma_1$ from U^- .

Without loss of generality, assume $i + 2 \leq j$ and use Theorem 3.2 to construct a CAT(-1)-piecewise ruled map Σ_1 which is Γ_1 -homotopic to S_1 . Because Σ_1 and S_1 are Γ_1 -homotopic, the surface Σ_1 also separates γ_i^+ and γ_j^+ from U^- . The surface Σ_1 must intersect $S \times [-1, i + 1]$ or it cannot separate γ_i^+ from U^- . Obviously $\chi(\Sigma_1) > \chi(S)$ and therefore by the Bounded Diameter Lemma the diameter of Σ_1 rel $N^{\leq \epsilon_0}$ is bounded from above by $D(\chi(S), \epsilon_0)$. Hence $\Sigma_1 \subset S \times (-2, i + 2)$. Since Σ_1 separates γ_j^+ from U^- , for every arc κ that connects $S \times \{-2\}$ and $S \times \{i + 2\}$, we have $\langle \Sigma_1, \kappa \rangle \neq 0$. As a result the projection of Σ_1 into $S \times \{0\}$ (given by the product structure on $S \times [-n - 1, n + 1]$) is a nonzero degree map. In particular $\chi(\Sigma_1) \leq \chi(S)$ which is a contradiction. \square

Remark 5.1. It might be useful to point out that the bounds for length and number of components of Γ all are computable.

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