Statement of Research

Hossein Namazi

My main research interests lie in the area of hyperbolic geometry, Kleinian groups and related areas such as Teichmüller theory, 3-dimensional topology and geometrical group theory. The majority of the work of the author is focused on the question of understanding the geometry of a hyperbolic 3-manifold. The recent confirmation of Thurston's Geometrization Conjecture suggests that a compact 3-manifold can be decomposed topologically into various pieces where each piece admits one of the so called "8 geometries". For seven of these geometries, there is a limited number of possibilities that admit such geometries and is well known how to describe and list these [Sc83]. What remains are the hyperbolic pieces for whom the question of describing the geometry of the hyperbolic metric is far from being answered completely. An interesting aspect of this question is understanding the interaction between the geometry of the hyperbolic manifold and various topological features of the 3-manifold. This is where most of the work of the author is concentrated at and we explain our major results below.

In some of the results below, we are concerned with a picture of the hyperbolic manifold that is "accurate" for all but finitely many elements in a category without being careful about how many examples we miss. These results are useful when one is dealing with infinitely many examples and is satisfied with descriptions that are possibly not quite effective or computable. There is however a different approach that we pursue in some of the work where we try to find effective means of describing the geometry. These results are usually more technical but they have the advantage of making it possible to compute some of the constants and bounds that are obtained.

In addition, many of the recent work of the author has been focused on finding "localized" descriptions of the geometry of a hyperbolic manifold. This is motivated by the expectation that most of the properties of the metric are local and should not depend on global assumption. We obtain some results in this direction and provide a few tools that can be used in order to find such local theorems. This is for example in contrast with the analytic approaches (such as Prelemann's proof of the geometrization) which require global assumptions and we believe this shows another advantage of our more constructive approach.

The Density Conjecture

One of the major conjectures in the study of Kleinian groups is the Bers-Thurston-Sullivan Density conjecture. The conjecture states that every finitely generated Kleinian groups is a limit of geometrically finite Kleinian groups. The proof of this conjecture is a culmination of many recent developments in the field. In particular the proof of the Tameness Conjecture by Agol [Ag] and Calegari-Gabai [CG06], the proof of the Ending Lamination Conjecture by Brock-Canary-Minsky

[ELC1, ELC2] and the convergence result of Kleineidam-Souto [KS02]. In the cases when the manifold has incompressible boundary, the mentioned results are sufficient to prove the conjecture. In [NS2], we prove the final required step in the general case. In addition we give a complete proof of the conjecture which simplifies the proof in some of the other cases as well. We prove our result in the setting of *pared manifolds* and therefore cover the cases that were missing from the previous attempts.

Assume $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ is a torsion free finitely generated Kleinian group. (It is not hard to extend the result to the case when Γ has torsion but to remain brief we ignore that case.) Following Thurston let $AH(\Gamma)$ be the set of conjugacy classes of discrete and faithful representations $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$. An element of $AH(\Gamma)$ is geometrically finite if for a representative $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ of that element, $\rho(\Gamma)$ has a fundamental domain with finitely many sides. We consider the space $AH(\Gamma)$ with the *algebraic topology*, i.e. a sequence $([\rho_i])_i$ in $AH(\Gamma)$ converges to $[\rho] \in AH(\Gamma)$ if there are representatives $\rho_i \in [\rho_i]$ and $\rho \in [\rho]$ so that for all $\gamma \in \Gamma$ the sequence $(\rho_i(\gamma))_i$ converges to $\rho(\gamma)$ in $\mathrm{PSL}_2(\mathbb{C})$. Now we can state the conjecture precisely.

Theorem 1 (Density Conjecture)**.** *If Γ is a finitely generated Kleinian group, the the set of geometrically finite points in $AH(\Gamma)$ is dense in the algebraic topology.*

Even more using well known results and in particular work of Sullivan and Marden, we have the following corollary.

Corollary 2. *If Γ is a finitely generated Kleinian group, then $AH(\Gamma)$ is the closure of its interior.*

It follows from Tameness that $N = \mathbb{H}^3/\Gamma$ is homeomorphic to the interior of a compact 3-manifold M. Even more if N^ϵ denotes the complement in N of all the ϵ-cusps (for an ϵ smaller than the Margulis constant) then it admits a *standard compact core* which can be identified with M. By this we mean a compact submanifold M whose complement is homeomorphic to a product and such that the inclusion of $P = M \cap \partial N^\epsilon$ into ∂N^ϵ is a homotopy equivalence. The pair (M, P) is a pard manifold. (See [Joh79] or [CMc04] for definition and an introduction.)

We denote by $AH(M, P)$ the subset of $AH(\Gamma)$ consisting of conjugacy classes of discrete and faithful representations of $\Gamma = \pi_1(M)$ into $\mathrm{PSL}_2(\mathbb{C})$ with the property that the image of every element whose conjugacy class is represented by a loop on P is a parabolic element. By construction ρ is a *minimally parabolic* element of $AH(M, P)$, i.e. the image of an element is parabolic if and only if its conjugacy class is represented by a loop in P. As a matter of fact we prove that ρ is a limit of geometrically finite, minimally parabolic elements of $AH(M, P)$.

The closure F of a component of $\partial M \setminus P$ is a compact surface which we call a *free side.* Obviously ends of N^ϵ are in one-to-one correspondence with free sides of (M, P). Suppose $\mathcal{E}$ is an end associated to the free side F. We say $\mathcal{E}$ is *convex cocompact* if it has a neighborhood whose intersection with the compact core of N is compact. In this case the geometry of the end is well understood by Ahlfors-Bers theory. In particular such an end faces a component of the quotient by Γ of the domain of discontinuity for the action of Γ on the boundary at infinity of $\mathbb{H}^3$. This quotient naturally gives a conformal structure on the interior of F which is called the *conformal structure at infinity.* The end invariant of a convex cocompact end is the point in the Teichmüller space of the free side F determined by this conformal structure. If $\mathcal{E}$ is not convex-cocompact, then we say it is *degenerate.* By work of Thurston and Canary [Can93b], every such end has an associated *ending lamination.* In other words, the geometry of $\mathcal{E}$ determines a filling geodesic lamination on F which we call the ending lamination. The end invariant of $\mathcal{E}$ in this case is its ending lamination.

Canary [Can93b] proved that the end invariants of N^ϵ satisfy the following two, rather mild, conditions:

(*) If M is an interval bundle over a compact (possibly unorientable) surface S and N has no convex-cocompact ends, then the projection of the ending lamination(s) to S is not a lamination.

(**) If a compressible free side F of ∂M faces a degenerate end $\mathcal{E}$, then the ending lamination is the support of a *Masur domain* lamination. Equivalently, the support of the ending lamination is not contained in the Hausdorff limit any sequence of meridians.

Note that by a *meridian* on F, we mean a simple noncontractible loop on F which is homotopically trivial in M.

We say a collection of end invariants for (M, P) is *filling* if it satisfies the above conditions. An immediate consequence of our proof is the following converse to the result of Canary mentioned above.

Theorem 3 (Namazi-Souto)**.** *Suppose (M, P) is a pared 3-manifold. Given a filling collection of end invariants for (M, P), there exists a minimally parabolic representation $\rho \in AH(M, P)$ and an embedding $(M, P) \to (N^\epsilon_\rho, \partial N^\epsilon_\rho)$ in the homotopy class determined ρ and for the hyperbolic manifold $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$, whose image is a standard compact core of N_ρ and such that the end invariants of N_ρ with respect to this standard core are the given end invariants in the beginning.*

It follows from the Ending Lamination Theorem, proved by Minsky [ELC1] and Brock-Canary-Minsky [ELC2], that the manifold N_ρ provided by the above theorem is unique up to isometry.

Using the same notation as above, assume F is a free side of (M, P) and λ is a geodesic lamination on F. We say λ is *realized* in N_ρ if there is a proper map $f : (F, \sigma) \to N_\rho$ which takes every leaf of λ to a geodesic in N_ρ and on the level of fundamental groups f induces the same map as the composition $F \hookrightarrow M \to N_\rho$. An important property of the ending lamination for an end of N^ϵ_ρ is that it is not realized [Can93b]. Our proof of the Density Conjecture basically boils down to proving that in some sense this property identifies the ending lamination. We state this result which is of its own interest.

Theorem 4 (Namazi-Souto)**.** *Let (M, P) be a pared manifold and $\rho : AH(M, P)$. Also let $(M', P') \subset (N^\epsilon_\rho, \partial N^\epsilon_\rho)$ be a relative compact core of $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$, and $\phi : (M, P) \to (M', P')$ in the homotopy class determined by ρ. Suppose λ is a filling Masur domain lamination on a free side F of (M, P) which is not realized in N_ρ. Then ϕ is homotopic, relative to the complement of a regular neighborhood of F, to a map $\phi_1 : (M, P) \to (M', P')$ such that:*

- *The restriction of ϕ_1 to F is a homeomorphism to some free side F' of (M', P').*

- *The end of N_ρ associated to F' is degenerate and has ending lamination $\phi_1(\lambda)$.*

We should also point out that even though the Density Conjecture is concerned only with the algebraic topology, our proof provides a convergence in the so called *strong topology*, i.e. the convergence is both geometric (in the Chabouty topology) and algebraic.

Theorem 5 (Namazi-Souto)**.** *Every finitely generated Kleinian group Γ is the algebraic limit of a sequence of discrete and faithful geometrically finite representations whose images converge also in the Chabauty topology to Γ. In other words, the geometrically finite representations are dense in the strong topology.*

Thurston's Uniform Injectivity Theorem

We prove a generalization of Thurston's Uniform Injectivity Theorem. Furthermore, this gives a new proof of Thurston's original statement and resolves a gap that existed in Thurston's original presentation. There are two advantages to our proof, one is that in our proof all the constants are computable and the proof is completely constructive. Thurston's original proof and other accounts and generalizations of this [Min92, Min00, Na2, Bow] all use geometric limit arguments and therefore do not provide computable constants. The second advantage is that our proof naturally generalizes to a broader situation compare to what was contained in the original statements. Suppose $f : S \to N$ is a map from a surface S equipped with a metric to a hyperbolic 3-manifold N and f preserves lengths of paths. We replace Thurston's *doubly incompressibility* assumption of f by a quantitive property that we call ϵ_0-*doubly incompressibility.* This requires doubly incompressibility only in a small neighborhood of the surface and therefore allows generalizing to the situations when for example we don't have a π_1-injective map. Also this requires only local assumptions on f and shows that it should be possible to extend the result to the cases when we don't want to make any assumption about parts of the manifold that are far from the surface.

On the other hand, we relax Thurston's assumption on requiring the surface to be a pleated surface and the metric on S to be hyperbolic. More or less, we are satisfied when (the metric structure on) S is CAT(-1) with an additional property that f *respects the thick-thin decomposition.* This last property is also quantified by a homeomorphism $\rho : [0, \infty) \to [0, \infty)$ and requires that for every $\epsilon > 0$, f maps the ϵ-thick part of S to the $\rho(\epsilon)$-thick part of N. These are obviously satisfied for pleated surfaces but our work allows other cases like simplicial hyperbolic surfaces and shrinkwrapped surfaces. In essence we reduce the assumption on the surface to a mild assumption that it is CAT(-1) and respects the thick-thin decomposition. One version of the theorem is the following:

Theorem 6 (Namazi-Souto)**.** *Suppose $f : S \to N$ is a simplicial hyperbolic surface realizing a lamination λ. If f is ϵ_0-doubly incompressible and respects the thick-thin decomposition for ρ, then for every $\epsilon_1 > 0$, there exists $\epsilon_2 > 0$ so that for every $x, y \in \lambda \cap S^{\geq \epsilon_0}$*

$$d_{\mathbb{P}N}(\mathbb{P}f(x), \mathbb{P}f(y)) \leq \epsilon_2 \quad \Rightarrow \quad d_S(x, y) \leq \epsilon_1.$$

As usual $\mathbb{P}N$ denotes the projective tangent bundle of N and $\mathbb{P}f$ is the lift of f from λ to this projective tangent bundle.

The importance of this result is in various applications of the theorem in the construction of models for the geometry of hyperbolic 3-manifolds. It allows using this result even when surfaces are not necessarily incompressible and makes it more likely to be able to find effective and computable constructions of the models.

Quasiconvexity and Shrinkwrapping

In [Na2] we use Calegari-Gabai's construction of shrinkwrapped surfaces [CG06] to improve a result of Minsky and provide various constructions that show the importance of these shrinkwrapped surfaces and their usefulness when only local assumptions are present. In fact, we rely more on the less analytic constructions of these maps due to Soma [So06, Ga]. Also we use an idea of Gabai to improve Soma's construction slightly and guarantee that the shrinkwrapped surfaces are simplicial hyperbolic as well. This relates these surfaces to simplicial hyperbolic surfaces that have been already in use for some time.

Given a hyperbolic 3-manifold N and $\eta > 0$, we say a geodesic link $\Gamma \subset N$ is η-separated if every arc of length $\leq 2\eta$ that connects two points of Γ is homotopic (rel. endpoints) to a subarc of Γ. Also assume $j : S \to N \setminus \Gamma$ is a map of a closed surface S into N. We say j is *2-incompressilble rel* Γ if the image of every closed curve is neither homotopically trivial in $N \setminus \Gamma$, nor is homotopic (in $N \setminus \Gamma$) to a multiple of a meridian of Γ. If α is a loop on S, by a representative of α in $N \setminus \Gamma$, we mean a loop in $N \setminus \Gamma$ which is freely homotopic (in $N \setminus \Gamma$) to $j(\alpha)$.

A large portion of the work in [Na2] is devoted to proving various technical results in constructing and describing shrinkwrapped surfaces. In particular we prove that if Γ is η-separated and a shrinkwrapped surface is 2-incompressible rel Γ, then the induced CAT(-1) structure admits a uniform thin-thick decomposition and the map into N respect the thin-thick decomposition for a homeomorphism $\rho : [0, \infty) \to [0, \infty)$ which only depends on ρ and the topology of the surface.

Our main theorem in the case of closed surfaces in [Na2] is the followoing:

Theorem 7. *Suppose $j : S \to N$ is 2-incompressible rel Γ, an η-separated geodesic link in the hyperbolic 3-manifold N. Also let $\mathcal{C}(B, j)$ denote the set of simple loops on S which have a representative of length $\leq B$ in $N \setminus \Gamma$. When B is large then $\mathcal{C}(B, j)$ is a K-quasiconvex subset of $\mathcal{C}(S)$ where K depends only on η, B and the topology of S.*

In fact, it is enough to assume B is bigger than the Bers' constant for S. We like to think of this theorem as a local version of Minsky's result in [Min01].

A natural question related to the above theorem is finding instances where one can guarantee the existence of η-separated geodesic links with respect to which a map $j : S \to N$ is 2-incompressible. We state a result that guarantees this in the presence of *wide product regions*. These are regions that are bilipschitz to a subset of a doubly degenerate surface group with large diameter relative to the thin part. The construction however has the potential of being used in a more general setting when for example, one does not exactly have this bilipschitz correspondence. Moreover we extend the result and consider embeddings $j : M \to N$ from a compact 3-manifold into N. We generalize the notion of 2-incompressibility for such maps and we prove the following. To simplify the statement, we assume M has no torus boundary and N has no parabolics.

Theorem 8. *Suppose $j : M \to N$ is an embedding of a compact 3-manifold M into a hyperbolic 3-manifold N with the property that the image of every component E of ∂M is either incompressible in M and $j(E)$ is incompressible to the outside or $j(E)$ is a level surface of a sufficiently wide product region. Then there exists a 0.025-separated geodesic link Γ inside the product regions, so that j is 2-incompressible rel Γ.*

We have obviously omitted some of the details regarding the quantifiers implemented in describing a wide product region. We should also point out the constants are all computable and the length of Γ and number of its components are all bounded and effective. The above two results can be used successfully in many instances when one can make local assumptions about the hyperbolic 3-manifold and obtain geometric descriptions of the metric. We should point out that the quasiconvexity of the set of simple loops with bounded length geodesic representatives is the basis of producing bilipschitz models that describe the geometry of the hyperbolic 3-manifold. Therefore we expect that we can use the above results and obtain local models for the geometry of various pieces of a hyperbolic 3-manifold. In addition effectivity of the construction together with the constructive proof of Thurston's Uniform Injectivity, explained above, should allow us to produce effective models with computable constants.

Bounded combinatorics and heegaard splittings

The results of [Na3] are restatements of the results in the author's dissertation [Na1]. These are however modified and the proofs are transformed as a result of applying results of [Na2, NS2, NS3]. We use shrinkwrapped surfaces and we rely heavily on their properties and construction that was mostly proven in [Na2].

A new feature of this work is that using our new version of the Uniform Injectivity Theorem [NS3] and Quasiconvexity of bounded length curves [Na2], we construct interpolations of ends of a hyperbolic structure quite effectively. Our emphasis is on hyperbolic structures on a handlebody but the construction is general.

To explain the main result briefly suppose H is a handlebody of genus > 1 and $R > 0$ a constant. We define a set of hyperbolic structures on H whose *end invariants* satisfy a combinatorial condition that we call *R-bounded combinatorics*. We prove this set is compact in a very strong sense which in particular means every sequence has a subsequence that converges both algebraically and geometrically to another structure of this form. We use this compactness and the interpolation that was explained above to show that these structures all have a uniformly bounded geometry, i.e. a uniform lower bound for the injectivity radii.

Now suppose N is one of these structures with R-bounded combinatorics on N. In the presence of this lower bound for the injectivity radii, the interpolation for the end of the hyperbolic structure becomes a controlled sweep-out of the geometry of the end of N. At this stage we use and idea of Mosher [Mo03] to prove that except for a (uniformly) bounded diameter compact core of N, the convex core of N is (uniformly) bilipschitz to a standard structure which is *the marked hyperbolic surface bundle* on a Teichmüller geodesic. This Teichmüller geodesic is contained in a thick subset of the Teichmüller space (independent of N) and is determined by the end invariant of N. This part requires a careful application of work of Mosher [Mo03] and Farb-Mosher [FM02]. We also explain and resolve a small technical gap that existed in the presentation of [FM02].

As a result for every hyperbolic structure N on H with R-bounded combinatorics we can provide a bilipschitz model for the geometry of the convex core where the bilipschitz constant depends on R and the genus of H. Also the model is described by the end invariant of N. The major application of this description is to study hyperbolic structures on a closed 3-manifold with a special Heegaard splitting. Given a constant $R > 0$ as before, we define a combinatorial condition for a Heegaard splitting that we again call *R-bounded combinatorics* and is obviously motivated by the definition for hyperbolic structures on a handlebody. Recall also the *Heegaard distance* for a Heegaard splitting which was defined by Hempel [He01] and measures the distance between the disk sets of the two handlebodies in the complex of curves of the Heegaard surface. We prove the following:

Theorem 9. *Given $\epsilon > 0$, if $M = H^+ \cup H^-$ is a Heegaard splitting with R-bounded combinatorics and sufficiently large Heegaard distance then M admits a complete negatively curved metric with sectional curvatures in $[-1-\epsilon, -1+\epsilon]$.*

In fact we prove much more than the mere existence of these metrics. Our construction is quite explicit using the hyperbolic structures with R-bounded combinatorics on a handlebody. This allows us to describe the geometry of the negatively curved manifold by a bilipschitz model with a bilipschitz constant that depends only on R and the genus of the splitting. Also such metrics admit a uniform lower bound for the injectivity radius and the metric is actually hyperbolic except in a set of bounded diameter.

It is possible to use a result of Tian [Ti90] and conclude that with the assumption of the above theorem, M is actually hyperbolic and the hyperbolic metric is close to the constructed metric. But we show how for most of the applications it is enough to work with the negatively metric obtained in the theorem. These applications are the same as our corollaries and applications in [NS1].

Bounded geometry and bounded combinatorics

A major tool for constructing 3-manifolds is to start from a certain number of pieces with boundary and construct a new manifold by gluing these pieces along paris of boundary components that are homeomorphic. This is obviously a very rich construction and perhaps too flexible. It is however interesting to ask how the choice of the original pieces and different types of gluings affects the geometry of the manifold that is obtained at the end.

In a joint work with Brock, Minsky and Souto, we try to explain a general setting in which gluings with bounded combinatorics are used on a family of atoroidal orientable compact 3-manifolds with boundary. Our main result is that all but finitely many of closed manifolds that are obtained in this way are hyperbolic with uniform bounded geometry and in fact the combinatorics of the gluings allow us to describe their geometries up to bilipschitz deformations by models that are obtained by fixed core pieces and models for the gluing regions that resemble canonical hyperbolic surface bundles over geodesics in a thick subset of the Teichmüller space.

A *decorated manifold* is a compact, oriented, irreducible and atoroidal 3-manifold M with no torus boundary components, equipped with a *complete marking* on its boundary which is the *deco-*

ration. (For a discussion of markings see [MM00].) The decoration is denoted by $\mu(M)$ and we use the notation $\mu(M, E)$ to represent the restriction of $\mu(M)$ to a component E of ∂M.

We say a marking τ (possibly a partial marking) on ∂M has *R-bounded combinatorics with respect to M* if:

1. For each component E of ∂M and a proper essential subsurface W of E,

   $$d_W(\mu(M, E), \tau|_E) \leq R,$$

   where $\tau|_E$ is the restriction of τ to E and $d_W(\cdot, \cdot)$ denotes the distance in $\mathcal{C}(W)$, the complex of curves of W, between projections to $\mathcal{C}(W)$. (Cf. [MM00].)

2. If E is a compressible component of ∂M,

   $$d_E(\mu(M, E), \tau|_E) \leq d_E(\Delta(E), \tau|_E) + R$$

   where $\Delta(E)$ is the set of all meridians on E as a subset of $\mathcal{C}(E)$.

3. If E is a cylindrical component of ∂M, $\tau|_E \neq \emptyset$, where by a cylindrical boundary component we mean one which is compressible or contains a boundary component of an essential, non boundary parallel, properly embedded annulus.

When M is an I-bundle over a surface (trivial or twisted), we need some other technical restrictions but we omit them to keep this exposition brief. For every component E of ∂M, we call $d_E(\mu(M, E), \tau_E)$ the *height of τ on E.*

Now suppose $\mathcal{M}$ is a fixed finite collection of decorated manifolds and Ξ is a set where every element is a copy of an element of $\mathcal{M}$. Also assume for every element $M \in \Xi$ and a boundary component E of ∂M, there is a unique element $M' \in \Xi$ with a unique boundary component E' (we allow $M = M'$ but $E \neq E'$) and there is an orientation reversing homeomorphism $\psi_E = \psi_{E'}^{-1} : E \to E'$. We can obtain a closed 3-manifold X as the identification space of elements of Ξ via these pairwise boundary identifications. In presence of such identifications, for every element $M \in \Xi$, we obtain a marking $\tau(M)$ on ∂M which restricted to a component E of ∂M is $\psi^{-1}(\mu(M', E'))$, where $\psi_E : E \to E'$ identifies E with boundary component E' of $M' \in \Xi$. We say this is a *gluing with R-bounded combinatorics* or an $(\mathcal{M}, R)$*-gluing* if for every $M \in \Xi$, $\tau(M)$ has R-bounded combinatorics with respect to M. The *height* of each identification $\psi_E : E \to E'$ for $E \subset \partial M$ and $E' \subset \partial M'$ is defined to be

$$\text{height}(\psi_E) = \text{height}(\psi_{E'}) = d_E(\mu(M, E), \psi_E^{-1}(\mu(M', E'))).$$

Given an $(\mathcal{M}, R)$-gluing X, we construct a model manifold which is homeomorphic to X and is equipped with a metric. This model is obtained completely combinatorially from pieces which are fixed metrics on elements of $\mathcal{M}$ and interval bundles whose geometry is determined by the gluing maps. The geometry of these interval bundles is combinatorially determined from the decorations and canonical hyperbolic surface bundles over geodesics in thick part of the Teichmüller spaces of the boundary components. Then we show the following theorem.

Theorem 10 (Brock-Minsky-Namazi-Souto)**.** *Given a collection* $\mathcal{M}$ *of decorated manifolds and* $R > 0$*, there exists* L *such that if* X *is an* $(\mathcal{M}, R)$*-gluing with sufficiently large heights then* X *is hyperbolic and with the hyperbolic metric it is* L*-bilipschitz to the described model for* X*.*

The above theorem is a culmination of various results in this direction where gluing with bounded combinatorics are allowed. Our approach however provides new elements that is worth mentioning. We allow decorated manifolds that are probably not acylindrical. This requires a convergence theorem that works in this setting. We prove an embedding theorem, where we start by a sequence of homomorphisms from the fundamental group of a decorated manifold into fundamental groups of a sequence of hyperbolic manifolds. We assume the restrictions to certain subgroups are *eventually* faithful and *respect* primitivity of elements. These completely algebraic conditions are however sufficient to guarantee a nice geometric embedding of the decorated manifold and appropriate product regions for its boundary components into all but finitely many of the hyperbolic manifolds in the beginning. Finally our arguments are very local in their nature. In particular, it should be possible to make the assumption on the height to depend only on the adjacent pieces. Also we can provide this construction when infinitely many pieces are used. This provides a hyperbolization theorem that seems to be beyond what the analytical methods can reach.

A special case of the above theorem is a class of Heegaard splittings that satisfy a combinatorial condition that we call *generalized* R*-bounded combinatorics.* As the name suggests, this generalizes our definition in [Na3]. The combinatorial condition for such splittings of genus g naturally rises to a decomposition of the 3-manifold into a finite number of possible decorated manifolds. Also the gluings will have R-bounded combinatorics. Therefore a corollary of the above theorem is the following:

Theorem 11 (Brock-Minsky-Namazi-Souto)**.** *Assume* $g > 1$ *is fixed. Given* R *there exists* $\eta > 0$ *and* L *such that all but finitely many closed 3-manifolds* M *with a genus* g *splitting with generalized* R*-bounded combinatorics admit a hyperbolic structure with injectivity radius* $\geq \eta$*. In addition the hyperbolic metric is* L*-bilipschitz to a model that can be described using the combinatorics of the splitting.*

Using different methods and especially geometric convergence of manifolds with uniform bounded geometry, we also obtain a converse to the above theorem:

Theorem 12 (Brock-Minsky-Namazi-Souto)**.** *Assume* $g > 1$ *is fixed. Given* $\eta > 0$ *there exists* R *such that if* M *is closed hyperbolic 3-manifold with a genus* g *splitting then this splitting satisfies the generalized* R*-bounded combinatorics condition.*

One can view the above two theorems as a fairly complete description of hyperbolic 3-manifolds with injectivity radius $> \eta$ and genus g splitting. But unfortunately the above two theorems are far from being effective and the constants are not computable. We use various algebraic and geometric convergences and at this stage, it seems to be quite hard to avoid these type of argument.

We should also point out a different but closely related approach of Biringer and Souto [BS] where they study closed hyperbolic 3-manifolds with uniform bounded geometry and a bounded number of generators for the fundamental group. They also obtain a decomposition into pieces that resembles the decomposition explained above. One difference between our approach and theirs is that they have no way of predicting the decomposition. In our setting however, we are more concerned with a description of the geometry that can be predicted combinatorially and topologically.

Unbounded combinatorics and hyperbolic structures

An important ingredient of our work in [Na1, Na3, NS1] and what we explained above is an assumption that we are dealing only with guings with bounded combinatorics. These combinatorial conditions are slightly different in different settings but they are more or less modifications of Minsky's conditions in [Min01]. Also they always correspond to uniform lower bounds for the injectivity radius. This last point shows that not only such assumptions are not general but they are not even "generic". Therefore one asks what if this condition is removed.

In another joint work with Brock, Minsky and Souto, we study this question and propose a way of bypassing this difficulty. This is obtained by an idea of "drilling and filling" whose main motivation lies in the fundamental work of Hodgson and Kerckhoff [HK05] and subsequent works of various other people and in particular Bromberg. The main idea is that if we have unbounded combinatorics, then combinatorially (using Minsky's analysis in the case of surface groups) we can predict which curve must be extremely short in the hyperbolic manifold. We drill an "unknotted" copy of the curve and hyperbolize the complement. Now we use the unbounded combinatorics in the drilled manifold. There is a slope on the new toroidal boundary component that a Dehn surgery on that slope brings us back to the original manifold. The unbounded combinatorics should guarantee that the loop associated to this slope is extremely long (with respect to a normalized length on the cusp boundary). Here is where work of Hodgson and Kerckhoff [HK05] can be used to guarantee that not only the manifold obtained by Dehn surgery on that slope is hyperbolic but the drilled loop will be extremely small and the geometry outside of the corresponding Margulis tube is close to the geometry outside of the rank 2 cusp in the drilled manifold.

We use this method successfully in a special setting and we prove what sometimes was referred to as Hempel's conejcture.

Theorem 13 (Brock-Minsky-Namazi-Souto). *Given $g \geq 2$, there exists a constant D_g such that if $M = H^+ \cup H^-$ is a genus g Heegaard splitting with Heegaard distance $\geq D_g$, then M is hyperbolic.*

We should point out that the above theorem easily follows from the Geometrization Conjecture but our proof is substantially different. More importantly it sheds more light on how the hyperbolic metric looks and gives various information that are missing when using the Geometrization.

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