Optimal liquidation with market parameter shift: a forward approach

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Almgren-Chriss model

- Price under temporary and permanent impact:
  
  \[ P_t = P_0 + \sigma W_t + \gamma (X_t - X_0) + \lambda \dot{X}_t \]

  - permanent impact
  - temporary impact

- Inventory process:
  
  \[ X_t = x - \int_0^t \xi_s ds, \quad \text{with} \quad X_T = 0 \]

- \( T \) liquidation time, \( 0 < T < \infty \)

- \( T \) chosen at \( t = 0 \)

- Parameters \( \sigma, \gamma, \lambda \) also assessed at \( t = 0 \)
Optimal liquidation problem (Schied et al.)

- Controlled state processes: inventory and revenue

\[
X_t^\xi := x - \int_0^t \xi_s ds
\]

\[
R_t^\xi := r + \sigma \int_0^t X_s^\xi dW_s - \lambda \int_0^t \xi_s^2 ds
\]

- Value function

\[
V(x, r, 0; T) := \sup_{\xi} \mathbb{E}\left[ -e^{-R_T^\xi} \mid X_0^\xi = x, R_0^\xi = r \right]
\]

with the liquidation terminal constraint

\[
\begin{align*}
\{ & v(0, r, T; T) = -e^{-r}, \\
& v(x, r, T; T) = -\infty, \quad \text{for } x \neq 0
\}
\end{align*}
\]
Solution under CARA utility

- Value function ($\sigma = 1$)

$$V(x, r, 0; T) = -\exp\left(-r + \frac{\sqrt{\lambda}}{2} x^2 \coth \left( \frac{T}{\sqrt{2\lambda}} \right) \right)$$

- Optimal trading rate for $0 \leq t \leq T$

$$\xi_t^* = -\frac{V_x(X_t^*, R_t^*, t; T)}{2\lambda V_r(X_t^*, R_t^*, t; T)} = \frac{1}{\sqrt{2\lambda}} X_t^* \coth \left( \frac{T - t}{\sqrt{2\lambda}} \right)$$

- Optimal inventory process

$$X_t^* = \frac{x \sinh \left( \frac{T - t}{\sqrt{2\lambda}} \right)}{\sinh \left( \frac{T}{\sqrt{2\lambda}} \right)}; \quad X_T^* = 0$$

- Model commitment: temporary price impact parameter $\lambda$

pre-chosen at $t = 0$, unrealistic!
Empirical studies point out to non-constant $\lambda$

- Liquidity profile not constant; intraday and intraweek patterns
- Small- and medium-capitalization stocks have liquidity profiles harder to predict/model (for longer horizons)
- Extreme events, e.g., Flash Crash (May 6, 2010); may need adaptive trading objectives that respond to what actually occurs as time enfolds

How can we then accommodate dynamic model change?
Classical approach

- Inflexibility w.r.t. model revision

\[(\lambda; \sigma, \gamma) \quad \text{pre-chosen model} \quad U_T(x, r)\]

\[t = 0 \quad \overset{T}{\rightarrow} \quad \text{backward construction} \]
Model commitment

▶ Dynamic Programming Principle

\[ V_{t,T}(\cdot) = \sup_{A_{[t,T]}} \mathbb{E}( U_T(X_T) | \mathcal{F}_t) = \sup_{A_{[t,s]}} \mathbb{E}( V_{s,T}(X_s) | \mathcal{F}_t) \]

▶ A two-period discrete time example

Model choice at \( t = 0 \):

\[
\begin{align*}
\lambda_1 & \quad \frac{T}{2} & \quad \lambda_2 & \quad T
\end{align*}
\]

Value functions:

\[
V_0(\cdot; \lambda_1, \lambda_2) \leftarrow V_{T/2}(\cdot; \lambda_2) \leftarrow U_T(\cdot)
\]

Dynamics/controls:

\[
R_0 \xrightarrow{\xi_1^*(\cdot; \lambda_1, \lambda_2)} R_{T/2}^\xi_1 \xrightarrow{\xi_2^*(\cdot; \lambda_2)} R_T^\xi_1^*, \xi_2^*
\]
Is model revision viable after we ”start”?

- At \( t = \frac{T}{2} \), suppose that model revision yields a new temporary price impact parameter \( \hat{\lambda}_2 \in \mathcal{F}_{\frac{T}{2}} \). The initially chosen \( \lambda_2 \) is not anymore valid.

- Two consequences
  1. Time-inconsistency for the second period given the new model

\[
\xi_2^*(\cdot ; \lambda_2) \neq \arg \max_{\xi_2} \mathbb{E} \left( U_T \left( X_{\frac{T}{2}}^{\xi_2} \right) \middle| \mathcal{F}_{\frac{T}{2}}, \hat{\lambda}_2 \right)
\]

2. \( \xi_1^* \) is still ”affected” since \( \lambda_2 \) is already embedded in \([0, \frac{T}{2}]\);
   \( \xi_1^* \) ”would not have been” optimal under the new model

\[
\xi_1^*(\cdot ; \lambda_1, \lambda_2) \neq \arg \max_{\xi_1} \mathbb{E} \left( V_{\frac{T}{2}} \left( X_{\frac{T}{2}}^{\xi_1}; \hat{\lambda}_2 \right) \middle| \mathcal{F}_0 \right)
\]
How do we then incorporate model revision, time-consistency, optimality and liquidation?

- Shall we commit at $t = 0$ to a specific model for many periods ahead?

- If yes, how do we then incorporate learning? Recall that the revised $\hat{\lambda}_2$ is $\mathcal{F}_{T/2}$-mble and not $\mathcal{F}_0$-mble.

- If we proceed "incrementally" forward in time, how do we then define optimality?

- What to do if we want to remain time-consistent?

- What are the consequences on (perfect) liquidation?
Forward performance approach (Musiela & Zariphopoulou, 2003, ...)

- Allows for **dynamic model revision** as market evolves
- It is built to maintain **time-consistency**
- Forward performance and portfolio processes **track** the market up to **current** time
- Existing forward results mainly assume **continuous model revision** and **continuous preference revision**
Liquidation under forward criteria

- Revision of price impact parameter \( \lambda \) may be done continuously or discretely.
- **Discrete** revision is more **realistic** and practical.
- When revision is done discretely, the \( \lambda \) is kept "**piece-wise constant**"

\[
\begin{align*}
\lambda_i & \in \mathcal{F}_{t_i} \\
\lambda_i & \in \mathcal{F}_{t_{i+1}}
\end{align*}
\]

- Shall we then "**align**" the frequency at which the forward performance process is built with the frequency the model is revised?
- After all, this is also the case in the **classical setting**!

Model chosen at \( t = 0 \), utility \( U_T \in \mathcal{F}_0 \)
A family of random functions and random times \( \{U_n, \tau_n\}_{n \geq 0} \) is a predictable forward performance process if

1. \( U_0 \) is a deterministic utility function with \( \tau_0 = 0 \), and \( U_n \in \mathcal{U}(\mathcal{F}_{\tau_{n-1}}), \tau_n \in \mathcal{F}_{\tau_{n-1}}, n \geq 1 \), where \( \mathcal{U}(\mathcal{F}_{\tau_{n-1}}) \) is the set of \( \mathcal{F}_{\tau_{n-1}} \)-measurable utility functions

2. For any admissible wealth process \( (X_t) \),

\[
U_{n-1}(X_{\tau_{n-1}}) \geq \mathbb{E}\left(U_n(X_{\tau_n}) | \mathcal{F}_{\tau_{n-1}}\right), \quad \text{for } n \geq 1
\]

3. There exists an admissible wealth process \( (X^*_t) \), such that

\[
U_{n-1}(X^*_{\tau_{n-1}}) = \mathbb{E}\left(U_n(X^*_\tau_{n}) | \mathcal{F}_{\tau_{n-1}}\right), \quad \text{for } n \geq 1
\]
Optimal liquidation, dynamic model revision, and forward approach

- Assess $\lambda_1$ for a small period ahead /“confidence time”, say for $(0, \tau_1]$
- Choose initial criterion at $t = 0$, solve forward problem in $(0, \tau_1]$
  \[ U_0(x, r, 0) \text{ given} \Rightarrow \text{find } U_1(x, r, \tau_1; \lambda_1) \]
- Assess $\lambda_2$ for the next liquidation period, $(\tau_1, \tau_2]$
  \[ U_1(x, r, \tau_1; \lambda_1) \text{ given} \Rightarrow \text{find } U_2(x, r, \tau_2; \lambda_1, \lambda_2) \]
- Continue this procedure

Therefore, one incorporates "learning" as the market evolves while maintaining time-consistency by pasting bit-by-bit single inverse liquidation problems.
Backward and forward approach

\[ \mathcal{M}_{[0,T]} \]  

Backward

\[ \mathcal{M}_{[0,\tau_1]} \quad \mathcal{M}_{(\tau_1,\tau_2]} \]  

Forward
Solving the "first" inverse liquidation problem

\[ U(r, x, 0) \]

- Find \( U(r, x, \tau_1) \in \mathcal{F}_0 \), s.t.

\[
U(r, x, 0) = \sup \mathbb{E}\left( U(R_{\tau_1}, X_{\tau_1}, \tau_1) \bigg| \mathcal{F}_0, \lambda_1 \right)
\]

\[
dX_t = -\xi_t \, dt, \quad dR_t = -\lambda_1 \xi_t^2 \, dt + X_t \, dW_t
\]

- What is a "reasonable choice" for the initial \( U(r, x, 0) \)?

Perhaps

\[
U(r, x, 0) \overset{?}{=} V(r, x, 0; T, \lambda_1),
\]

but the admissible class of \( U(r, x, 0) \) is actually bigger
Main result: solution of the first problem

Suppose at time $\tau_0 = 0$, the initial utility is $U_0(r, x) = -e^{-r+g(x)}$. Furthermore, assume that there exist positive constants $a \geq b > 0$, such that for any $x > 0$, the function $g(x)$ satisfies

$$g''(x) \leq a \quad \text{and} \quad \frac{g'(x)}{x} \geq b$$

Then, the forward problem is well posed in the sense that an optimal admissible liquidation strategy exists for all time $t$, $0 \leq t < T^g(\lambda)$, where

$$T^g(\lambda) := \sqrt{2\lambda} \min \left\{ \tanh^{-1} \left( \frac{b}{\sqrt{2\lambda}} \wedge 1 \right), \coth^{-1} \left( \frac{a}{\sqrt{2\lambda}} \vee 1 \right) \right\}$$

$(\tanh^{-1}(1) = \coth^{-1}(1) = \infty)$
Classical existing result is a special case of the "single-period" forward problem

Choose

\[ U_0(r, x) = V(r, x, 0; T) \]

or

\[ U_0(r, x) = V(r, x, 0; \infty), \]

then the two problems reconcile. In particular,

\[ T^g(\lambda) = \text{liquidation time} = T \]

The backward case is a special case of the forward by properly choosing the initial condition
General forward solution

- For each model revision period \([\tau_n, \tau_{n+1}]\), solve a "single-period" forward problem with random coefficients

\[
\lambda_{n+1} \in \mathcal{F}_{\tau_n}
\]

\[
\tau_n \quad \text{and} \quad \tau_{n+1}
\]

\[
U_{\tau_n} \in \mathcal{F}_{\tau_{n-1}} \quad \text{and} \quad U_{\tau_{n+1}} \in \mathcal{F}_{\tau_n}
\]

- Serious issues with ill-posedness
Need to solve (period-by-period) a random ill-posed HJB

\[
U_t(x, r, t; \omega) + \frac{1}{2}x^2 U_{rr}(x, r, t; \omega) - \min_\xi \left( \lambda_{n+1}(\omega) U_r(x, r, t; \omega) \xi^2 + U_x(x, r, t; \omega) \xi \right) = 0, \quad \tau_n \leq t < \tau_n + 1
\]

- Inverse problem: given \(U(x, r, \tau_n; \omega) \in \mathcal{F}_{\tau_{n-1}}\), find

\[
U(x, r, \tau_{n+1}; \omega) \in \mathcal{F}_{\tau_n}
\]
Solution of the forward problem

- The predictable forward performance process at $\tau_n$ depends on the realized market model up to $\tau_n$, rather than model dynamics beyond $\tau_n$ as in the classical case

$$U(r, x, \tau_n) = -e^{-r} + \alpha_n(\lambda_1, \cdots, \lambda_n)x^2 \in F_{\tau_n-1}$$

- The quadratic coefficients $\alpha'_n$ can be computed recursively forward in time rather than through backward recursion as in the classical case

$$\alpha_n = \frac{\alpha_{n-1} \cosh \left( \frac{2(\tau_n - \tau_{n-1})}{\sqrt{2}\lambda_{n-1}} \right) - \left( \frac{\sqrt{2}\lambda_{n-1}}{4} + \frac{\alpha_{n-1}^2}{\sqrt{2}\lambda_{n-1}} \right) \sinh \left( \frac{2(\tau_n - \tau_{n-1})}{\sqrt{2}\lambda_{n-1}} \right)}{\cosh \left( \frac{\tau_n - \tau_{n-1}}{\sqrt{2}\lambda_{n-1}} \right) - \alpha_{n-1} \sqrt{\frac{2}{\lambda_{n-1}}} \sinh \left( \frac{\tau_n - \tau_{n-1}}{\sqrt{2}\lambda_{n-1}} \right)}^2,$$

with $\alpha_1(\lambda_1) = \sqrt{\frac{\lambda_1}{2}} \coth \left( \frac{\tau_1}{\sqrt{2}\lambda_1} \right)$
Reconcile with the classical results

If $\lambda_1 = \lambda_2 = \cdots \equiv \lambda$, i.e., constant price impact profile, then

- $(\tau_n)_{n \geq 0}$ is deterministic and $\tau_n \uparrow T$, as $n \to \infty$
- $U(r, x, \tau_n) = V(r, x, \tau_n; T)$, for $n \geq 0$
- Forward optimal strategy identical to the classical one
- In particular, liquidation completes exactly at $T$
When do we fully liquidate under model revision?

- Do we liquidate **exactly** at T?

- **Earlier** or **later** than T?

- How do we balance accurate model revision with (preassigned) liquidation time T?

- The **classical liquidation** policy is **inaccurate**; because the initial model has by now **changed**!
Trade-off

- **Classical case**
  - Perfect liquidation at $T$
  - wrong model
  - wrong value function

- **Forward case**
  - Liquidation time may be close to $T$
    (depending on model fluctuations)
  - accurately revised model
  - preservation of optimality
Forward liquidation and classical liquidation time $T$

- Depending on the fluctuations of price impact, we may liquidate before or after $T$, or even at $T$.

- One of the results:

\[
\begin{align*}
    &\lambda_n \quad X_{\tau_n}^* = 0 \\
    &\tau_{n-1} \quad \tau_n \quad T
\end{align*}
\]

Early liquidation occurs if $n$-th period occurs before $T$ and

\[
2\alpha_{n-1}(\lambda_1, \cdots, \lambda_{n-1}) > \sqrt{2\lambda_n} \quad \text{a.s.}
\]

i.e., full liquidation can be achieved if price impact is small.
Conclusions

- **Classical** approach requires full model specification at $t = 0$

- **Forward** approach is flexible to track the market, revising trading targets (e.g. trading horizon and volume) as market evolves

- Single-horizon and multiple-horizon formulations generalize existing optimal liquidation results under CARA utility

- Robust convergence results to the continuous time case
References


References


Thank you.