Optimal liquidation with market parameter shift: a forward approach

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Almgren-Chriss model

- Price under temporary and permanent impact

\[ P_t = P_0 + \sigma W_t + \gamma (X_t - X_0) + \lambda \dot{X}_t \]

- Inventory process: \( X_t = x - \int_0^t \xi_s ds \), with \( X_T = 0 \)

- \( T \) liquidation time, \( 0 < T < \infty \)

- Parameters \( \sigma, \gamma, \lambda \) assessed at \( t = 0 \); \( T \) also chosen at \( t = 0 \)
Optimal liquidation problem (Schied et al.)

- Controlled state processes: inventory and revenue

\[ X_t^\xi := x - \int_0^t \xi_s ds \]

\[ R_t^\xi := r + \sigma \int_0^t X_s^\xi dW_s - \lambda \int_0^t \xi_s^2 ds \]

- Value function

\[ V(x, r, 0; T) := \sup_{\xi} \mathbb{E} \left[ - e^{-R_T^\xi} \mid X_0^\xi = x, R_0^\xi = r \right] \]

with the liquidation terminal constraint

\[
\begin{cases}
  v(0, r, T; T) = -e^{-r}, \\
  v(x, r, T; T) = -\infty, \quad \text{for } x \neq 0
\end{cases}
\]
Solution under CARA utility

- Value function ($\sigma = 1$)
  \[
  V(x, r, 0; T) = -\exp \left( -r + \sqrt{\frac{\lambda}{2}} x^2 \coth \left( \frac{T}{\sqrt{2\lambda}} \right) \right)
  \]

- Optimal trading rate for $0 \leq t \leq T$
  \[
  \xi_t^* = -\frac{V_x(X_t^*, R_t^*, t; T)}{2\lambda V_r(X_t^*, R_t^*, t; T)} = \frac{1}{\sqrt{2\lambda}} X_t^* \coth \left( \frac{T - t}{\sqrt{2\lambda}} \right)
  \]

- Optimal inventory process
  \[
  X_t^* = \frac{x \sinh \left( \frac{T-t}{\sqrt{2\lambda}} \right)}{\sinh \left( \frac{T}{\sqrt{2\lambda}} \right)}; \quad X_T^* = 0
  \]

- Model commitment: temporary price impact parameter $\lambda$
  chosen at $t = 0$, unrealistic!
Empirical studies point out to non-constant $\lambda$

- Liquidity profile not constant; intraday and intraweek patterns
- Small- and medium-capitalization stocks have liquidity profiles harder to predict/model (for longer horizons)
- Extreme events, e.g., Flash Crash (May 6, 2010); may need adaptive trading objectives that respond to what actually occurs
Classical approach

- Model Commitment

\[ (\lambda; \sigma, \gamma) \quad \text{no model flexibility} \quad U_T(x, r) \]

\[ t = 0 \quad \text{DPP} \quad T \]

Backward construction
Model commitment

▶ Dynamic Programming Principle

\[ V_{t,T}(x) = \sup_{A_{[t,T]}} \mathbb{E} \left( U_T(X_T) \mid \mathcal{F}_t \right) = \sup_{A_{[t,s]}} \mathbb{E} \left( V_{s,T}(X_s) \mid \mathcal{F}_t \right) \]

▶ A two-period discrete time example

Model choice at \( t = 0 \): \[ 0 \quad \lambda_1 \quad \frac{T}{2} \quad \lambda_2 \quad T \]

Value functions: \[ V_0(\cdot ; \lambda_1, \lambda_2) \leftarrow V_{\frac{T}{2}}(\cdot ; \lambda_2) \leftarrow U_T(\cdot) \]

Dynamics/controls: \[ R_0 \xrightarrow{\delta_1^*(\cdot ; \lambda_1, \lambda_2)} R_{\frac{T}{2}}^{\delta_1^*} \xrightarrow{\delta_2^*(\cdot ; \lambda_2)} R_T^{\delta_1^*, \delta_2^*} \]
Is model revision viable after we "start" \( t > 0 \)?

- At \( t = \frac{T}{2} \), suppose that model revision yields \( \hat{\lambda}_2 \in \mathcal{F}_{\frac{T}{2}} \) and not \( \lambda_2 \)
- Two consequences
  
  1. Time-inconsistency for the second period given the new model

\[
\delta_2^*(\cdot; \lambda_2) \neq \arg \max_{\delta_2} E_{\mathbb{P}} \left( U_T \left( X_{\frac{T}{2}}^{\delta_2} \right) \mid \mathcal{F}_{\frac{T}{2}}, \hat{\lambda}_2 \right)
\]

2. \( \delta_1^* \) is "polluted" since \( \lambda_2 \) is already embedded in \([0, \frac{T}{2}]\);
   \( \delta_1^* \) "would not have been" optimal under the new model

\[
\delta_1^*(\cdot; \lambda_1, \lambda_2) \neq \arg \max_{\delta_1} E_{\mathbb{P}} \left( V_{\frac{T}{2}} \left( X_{\frac{T}{2}}^{\delta_1}; \hat{\lambda}_2 \right) \mid \mathcal{F}_0 \right)
\]
How do we then incorporate model revision, time-consistency and optimality?

- Shall we commit at $t = 0$ to a specific model for many periods ahead?

- If yes, how do we then incorporate learning? Recall that the revised $\hat{\lambda}_2$ is $\mathcal{F}_T^{\frac{T}{2}}$-mble and not $\mathcal{F}_0$-mble.

- If we proceed "incrementally" forward in time, how do we then define optimality?

- What to do if we want to remain time-consistent?
Forward performance approach (Musiela & Zariphopoulou, 2003, ...)

- Allows for **dynamic model revision** as market evolves
- It is built to maintain **time-consistency**
- Forward performance process and portfolio **adapt** to current information
- Existing forward results mainly assume **continuous model revision** and **continuous preference revision**
Liquidation under forward criteria

- Revision of price impact parameter $\lambda$ may be done continuously or discretely.
- Discrete revision is more realistic and practical.
- When revision is done discretely, the $\lambda$ is kept "piece-wise constant"

$$\lambda_i$$

- $\lambda_i \in \mathcal{F}_{t_i}$

- Shall we then "align" the frequency at which the forward performance process is built with the frequency the model is revised?
- After all, this is the classical case!

$\text{Model chosen at } t = 0, \text{ utility } U_T \in \mathcal{F}_0$

A family of random functions and random times $\{U_n, \tau_n\}_{n\geq 0}$ is a predictable forward performance process if

1. $U_0$ is a deterministic utility function with $\tau_0 = 0,$ and $U_n \in \mathcal{U}(\mathcal{F}_{\tau_{n-1}})$, $\tau_n \in \mathcal{F}_{\tau_{n-1}}$, $n \geq 1$, where $\mathcal{U}(\mathcal{F}_{\tau_{n-1}})$ is the set of $\mathcal{F}_{\tau_{n-1}}$-measurable utility functions

2. For any admissible wealth process $(X_t)$,

$$U_{n-1}(X_{\tau_{n-1}}) \geq \mathbb{E}\left( U_n(X_{\tau_n}) \mid \mathcal{F}_{\tau_{n-1}} \right), \quad \text{for } n \geq 1$$

3. There exists an admissible wealth process $(X^*_t)$, such that

$$U_{n-1}(X^*_{\tau_{n-1}}) = \mathbb{E}\left( U_n(X^*_{\tau_n}) \mid \mathcal{F}_{\tau_{n-1}} \right), \quad \text{for } n \geq 1$$
Optimal liquidation, dynamic model revision, and forward approach

- **Assess** \( \lambda_1 \) for a small period ahead /“confidence time”, say for \((0, \tau_1]\)
- Choose initial criterion at \(t = 0\), solve **forward** problem in \((0, \tau_1]\)

\[
U_0(x, r, 0) \text{ given } \implies \text{find } U_1(x, r, \tau_1; \lambda_1)
\]

- **Assess** \( \lambda_2 \) for the next liquidation period, \((\tau_1, \tau_2]\)

\[
U_1(x, r, \tau_1; \lambda_1) \text{ given } \implies \text{find } U_2(x, r, \tau_2; \lambda_1, \lambda_2)
\]

- Continue this procedure

Therefore, one **incorporates ”learning”** as market evolves while maintaining **time-consistency** by pasting **bit-by-bit** single inverse liquidation problems.
Solving the "first" inverse liquidation problem

\[ U(r, x, 0) \]

- Find \( U(r, x, \tau_1) \), s.t.

\[
U(r, x, 0) = \sup \mathbb{E} \left( U(R_{\tau_1}, X_{\tau_1}, \tau_1) \middle| \mathcal{F}_0, \lambda_1 \right)
\]

\[
dX_t = -\xi_t dt, \quad dR_t = -\lambda_1 \xi_t^2 dt + X_t dW_t
\]

- What is a "reasonable choice" for the initial \( U(r, x, 0) \)?

Perhaps

\[
U(r, x, 0) = V(r, x, 0; T, \lambda_1),
\]

but the admissible class of \( U(r, x, 0) \) is actually bigger.
Main result: solution of the first problem

Suppose at time $\tau_0 = 0$, the initial utility is $U_0(r, x) = -e^{-r + g(x)}$. Furthermore, assume that there exist constants $a \geq b > 0$, such that for any $x > 0$, the function $g(x)$ satisfies

$$g''(x) \leq a \quad \text{and} \quad \frac{g'(x)}{x} \geq b.$$ 

Then, the forward problem is well posed in the sense that an optimal admissible pure liquidation strategy exists for time $0 \leq t < T^g(\lambda)$, where the explosion time

$$T^g(\lambda) := \sqrt{2\lambda} \min \left\{ \tanh^{-1} \left( \frac{b}{\sqrt{2\lambda}} \wedge 1 \right), \coth^{-1} \left( \frac{a}{\sqrt{2\lambda}} \vee 1 \right) \right\}$$

$(\tanh^{-1}(1) = \coth^{-1}(1) = \infty)$
Classical existing result is a special case of the "single-period" forward problem

- Choose

\[ U_0(r, x) = V(r, x, 0; T) \]

or

\[ U_0(r, x) = V(r, x, 0; \infty), \]

then the two problems reconcile. In particular,

\[ T^g(\lambda) = \text{explosion time} = \text{liquidation time} = T \]

- The backward case is a special case of the forward by properly choosing the initial condition
General forward solution

- For each model revision period \([\tau_n, \tau_{n+1}]\), solve a "single-period" forward problem with random coefficients
  \[
  \lambda_{n+1} \in \mathcal{F}_{\tau_n}
  \]
  \[
  \tau_n \quad \tau_{n+1}
  \]

  \[
  U_{\tau_n} \quad \rightarrow \quad U_{\tau_{n+1}}
  \]

- Serious issues with ill-posedness
  Need to solve (period-by-period) a random ill-posed HJB

  \[
  U_t(x, r, t; \omega) + \frac{1}{2} x^2 U_{rr}(x, r, t; \omega) - \min_{\xi} \left( \lambda_{n+1}(\omega) U_r(x, r, t; \omega) \xi^2 \right.
  \]

  \[
  + U_x(x, r, t; \omega) \xi \right) = 0, \quad \tau_n \leq t < \tau_{n+1}
  \]

- Inverse problem: given \(U(x, r, \tau_n; \omega)\), find \(U(x, r, \tau_{n+1}; \omega)\)
Example

- To compare with the classical case, choose the initial datum to coincide with the value function (Schied et al.)

\[ U_0(r, x) = -e^{-r + \sqrt{\frac{\lambda_1}{2}} \coth \left( \frac{T}{\sqrt{2\lambda_1}} \right)} x^2 \]

- Solve the sequential single-period inverse problems
- Monitor "model reassessment times" through "explosion times"
- A sufficient condition for choosing reassessment times is

\[ \tau_n - \tau_{n-1} < T^g(\lambda_1, \cdots, \lambda_n) := \sqrt{2\lambda_n} \min \left\{ \tanh^{-1} \left[ \frac{2\alpha_{n-1}(\lambda_1, \cdots, \lambda_{n-1})}{\sqrt{2\lambda_n}} \land 1 \right], \coth^{-1} \left[ \frac{2\alpha_{n-1}(\lambda_1, \cdots, \lambda_{n-1})}{\sqrt{2\lambda_n}} \lor 1 \right] \right\} \in F_{\tau_{n-1}} \]
Solution of the forward problem

The predictable forward performance process at $\tau_n$ turns out to be

$$U_n(r, x) = -e^{-r + \alpha_n(\lambda_1, \ldots, \lambda_n)}x^2 \in F_{\tau_{n-1}},$$

where the quadratic coefficients $\alpha'_n$'s can be computed recursively,

$$\alpha_n = \frac{\alpha_{n-1} \cosh \left( \frac{2(\tau_n - \tau_{n-1})}{\sqrt{2\lambda_{n-1}}} \right) - \left( \frac{\sqrt{2\lambda_{n-1}}}{4} + \frac{\alpha_{n-1}^2}{\sqrt{2\lambda_{n-1}}} \right) \sinh \left( \frac{2(\tau_n - \tau_{n-1})}{\sqrt{2\lambda_{n-1}}} \right)}{\cosh \left( \frac{\tau_n - \tau_{n-1}}{\sqrt{2\lambda_{n-1}}} \right) - \alpha_{n-1} \sqrt{2\lambda_{n-1}} \sinh \left( \frac{\tau_n - \tau_{n-1}}{\sqrt{2\lambda_{n-1}}} \right)}^2,$$

with $\alpha_1(\lambda_1) = \sqrt{\frac{\lambda_1}{2}} \coth \left( \frac{\tau_1}{\sqrt{2\lambda_1}} \right)$
Reconcile with the classical results

If \( \lambda_1 = \lambda_2 = \cdots \equiv \lambda \), i.e., constant price impact profile, then

- \((\tau_n)_{n \geq 0}\) is deterministic and \( \tau_n \nearrow T \), as \( n \to \infty \)
- \( U_n(r, x) = V(r, x, \tau_n; T) \), for \( n \geq 0 \)
- Forward optimal strategy identical to the classical one
- In particular, liquidation completes exactly at \( T \)
When do we fully liquidate under model revision?

- Do we liquidate \textit{exactly} at $T$?

- Earlier or later than $T$?

- How do we balance accurate model revision with (preassigned) liquidation time $T$?

- The classical liquidation policy is \textit{inaccurate}; because the initial model has by now \textit{changed}!
Trade-off

- **Classical case**
  
  Perfect liquidation at T, wrong model, wrong value function

- **Forward case**
  
  Liquidation time may be close to T (depending on model fluctuations), accurately revised model, preservation of optimality
Forward liquidation and classical liquidation time $T$

- Depending on the fluctuations of price impact, we may liquidate before or after $T$, or even at $T$.

- For example,

\[ \lambda_n X^*_{\tau_n} = 0 \]

Early liquidation occurs if $n$-th period occurs before $T$ and

\[ 2\alpha_{n-1}(\lambda_1, \cdots, \lambda_{n-1}) > \sqrt{2\lambda_n} \quad \text{a.s.} \]

i.e., full liquidation can be achieved if price impact is small.
Going from discrete to continuous model revision

- We can show that under mild conditions, the predictable forward process converges to a limit that can be identified with the so-called "zero-volatility" forward (MZ 2010)

- Therefore, we can accommodate infinitesimal changes as the market evolves
Continuously updated market impact parameter

- Continuous time forward performance process

Given \((\Omega, \mathcal{F}, \mathbb{P})\) with Brownian filtration \(\mathcal{F}_t^W\), assume

\[
\begin{aligned}
    dU(x, r, t) &= b(x, r, t)\,dt + a(x, r, t)\,dW_t \\
    dX_t &= -\xi_t\,dt \\
    dR_t &= \sigma_t X_t\,dW_t - \lambda_t \xi_t^2\,dt
\end{aligned}
\]

- SPDE for the forward performance process is

\[
    dU(x, r, t) = \left( -\frac{U_x(x, r, t)^2}{4\lambda_t U_r(x, r, t)} - \left( \frac{\sigma_t}{2} U_{rr}(x, r, t) + a_r(x, r, t) \right) \sigma_t x^2 \right) dt + a(x, r, t)\,dW_t
\]
Solution of the SPDE for continuously changing market impact

- Set \( a(x, r, t) \equiv 0 \), the SPDE reduce to

\[
U_t(x, r, t) + \frac{U_x(x, r, t)^2}{4\lambda_t U_r(x, r, t)} + \frac{1}{2} U_{rr}(x, r, t)\sigma_t^2 x^2 = 0
\]

- Then, the forward process is given by

\[
U(x, r, t) = -e^{-r+x^2\alpha(t)}
\]

with \( \alpha(t) \) solving the random ODE

\[
\frac{d\alpha(t)}{dt} = \frac{\alpha^2(t)}{\lambda_t} - \frac{\sigma_t^2}{2}
\]
Forward coordinated variation case

Suppose that the market coefficients satisfy the coordinated variation condition (Almgren 2012)

$$\sigma_t^2 \lambda_t = 1, \forall t > 0 \text{a.s.}$$

In classical case, $\sigma_t$ and $\lambda_t$ pre-chosen at $t = 0$. No need to do this in the forward case!

For example, if we choose

$$U(x, r, 0) = -e^{-r + \frac{x^2}{\sqrt{2}}} \coth \left( \frac{T}{\sqrt{2}} \right)$$

Then, for $0 < t < \tau := \inf \{ s > 0 | \int_0^s \frac{1}{\lambda_u} du = T \}$,

$$U(x, r, t) = -e^{-r + \frac{x^2}{\sqrt{2}}} \coth \left( \frac{T - \int_0^t \frac{1}{\lambda_s} ds}{\sqrt{2}} \right)$$

is the unique (zero volatility) forward performance process with $\lambda_s$ being obtained forward in time.
Forward coordinated variation case

The optimal admissible inventory process is

\[ X^*_t = xe^{-\int_0^t \frac{1}{\sqrt{2}\lambda_s} \coth\left( \frac{T-\int_0^s \frac{1}{\lambda_u} du}{\sqrt{2}} \right) ds} \]

- If market impact is large, e.g.,

\[ \int_0^t \frac{1}{\lambda_s} ds < T - \epsilon, \forall t > 0 \ a.s. \]

then \( X^*_t > C > 0, \forall t > 0 \ a.s. \)

- If market impact is moderate, e.g., \( 0 < C_1 < \lambda_t < C_2 \)

uniformly in \((t, \omega)\), then \( X^*_T = 0, \ a.s. \) for \( \int_0^T \frac{1}{\lambda_s} ds = T \)
Convergence

- Consider the sequence of \( \{0 = \tau_0^N < \tau_1^N < \tau_2^N < \cdots\} \) with vanishing maximum partition as \( N \to \infty \)

- Define the linear interpolation of the mapping \( \tau_n^N \mapsto \alpha_n^N \), with \( \alpha_n^N \) from the multiple-horizon updating procedure

- Let \( \alpha_0^N \equiv \sqrt{\frac{\lambda_0}{2}} \coth \left( \frac{T}{\sqrt{2\lambda_0}} \right) \)

- Assumption 1: \( \lambda_t \) and \( \sigma_t \) are left-continuous with the property \( \inf_{t \geq 0} \lambda_t > 0 \) and \( 0 < \inf_{t \geq 0} \sigma_t < \sup_{t \geq 0} \sigma_t < \infty \) a.s.
Convergence

There exists $t^* > 0$ and a continuous function $\alpha : [0, t^*) \to \mathbb{R}$, s.t., for any $t \in [0, t^*)$,

$$\lim_{N \to \infty} \sup_{s \in [0, t]} |\alpha_N(s) - \alpha(s)| = 0, \text{ a.s.}$$

Moreover, $\alpha$ and $t^*$ are uniquely determined by the following conditions:

$$\alpha(0) = \sqrt{\frac{\lambda_0}{2}} \coth \left( \frac{T}{\sqrt{2\lambda_0}} \right), \quad \frac{d\alpha(t)}{dt} = \frac{\alpha^2(t)}{\lambda_t} - \frac{\sigma_t^2}{2}, \text{ a.s.} \quad (1)$$

for $t \in [0, t^*)$, and

$$t^* = \sup\{s > 0 : \text{there exists a bounded solution to (1) for } t \in [0, s]\}$$
Numerical Example

Figure 1: Trading strategies under various price impact profiles
Conclusion

- Classical approach requires full model specification at $t = 0$

- Forward approach is flexible to track the market, revising trading targets (e.g. trading horizon and volume) as market evolves;

- Single-horizon and multiple-horizon formulation generalize existing optimal liquidation results under CARA utility

- Robust convergence results to the continuous time case
References


References


Thank you.