The James and James Tree spaces

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Introduction

The main focus of this thesis is the James and James Tree spaces. These are two Banach spaces with very interesting but pathological properties. James Space was originally constructed by R.C. James [1] and served as the first example of a non-reflexive Banach space which coincides with its second conjugate. James Tree Space was constructed almost twenty years later by R.C. James [2], as a negative answer to S. Banach conjecture that each space with non-separable conjugate needs to contain an isomorphic copy of $\ell_1$. Let us give a brief outline of this thesis.

In the first chapter we introduce the notion of a Schauder basis in a Banach space. Schauder basis is a countable topological basis, meaning it strongly relies on the analytical structure given by the norm. In contrast to the algebraic basis, the existence of a Schauder basis is not guaranteed by the Axiom of Choice.

In the second chapter, we define a space similar to the original James space defined by R.C. James and we denote it as $J$. The space $J$ is ”almost” reflexive and does not contain neither $\ell_1$ nor $c_0$. We also define the classical James Space, which we denote by $J'$ and equip it with two equivalent norms. We are showing $J'$ is isomorphic with $J$ and that it is isometric to its second conjugate with respect to one of the norms defined. Therefore, $J$ might not be reflexive, but it is isomorphic to its second conjugate. We finally show that $J$ is $\ell_2$-saturated, meaning every weakly null sequence in $J$ has subsequence equivalent to the standard basis of $\ell_2$. We conclude that each infinite dimensional subspace of $J$ contains an isomorphic copy of $\ell_2$.

In the third chapter, we define the James Tree space and denote it as $JT$. As mentioned above $JT$ was constructed by R.C. James as a counterexample to the conjecture that every Banach space with non-separable dual contains $\ell_1$. This construction is possible using a binary tree basis. This is actually the main difference between $J$ and $JT$. Intuitively, each branch of $JT$ is
James space. Consequently $\mathcal{JT}$ inherits many of the properties $\mathcal{J}$ has but it also has a much richer structure since it consists of uncountably many James spaces.
Chapter 1

Schauder Bases

Throughout this manuscript, all Banach spaces considered will be infinite dimensional unless stated. Moreover we will denote $B_X$ the unit ball of $X$ and $S_X$, the unit sphere. Finally given a sequence $\{e_n\}_{n \in \mathbb{N}}$ in $X$ we will be writing

$$<e_n : n \in \mathbb{N} > = \text{span} \{e_n : n \in \mathbb{N} \}, \quad [e_n : n \in \mathbb{N} ] = <e_n : n \in \mathbb{N} >,$$

where the closure is taken with the respect to the norm of $X$.

Finally given a sequence $\{x_n\}_{n \in \mathbb{N}}$ we define its support as

$$\text{supp } x_n = \{n \in \mathbb{N} : x_n \neq 0 \}.$$

Similarly if $X \neq \emptyset$ and $f : X \to \mathbb{R}$ we define its support as

$$\text{supp } f = \{x \in X : f(x) \neq 0 \}.$$

1.1 Definition of a Schauder basis

Definition 1.1.1. Let $X$ be a Banach space and $\{e_n\}_{n \in \mathbb{N}}$ a sequence of distinct elements of $X$. The sequence $\{e_n\}_{n \in \mathbb{N}}$ is called a Schauder basis, or just a basis, of $X$ if for any $x \in X$ there is a unique sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ such that

$$x = \sum_{n=1}^{\infty} \lambda_n e_n.$$

Remark 1.1.1. Let $X$ be a Banach space with a basis $\{e_n\}_{n \in \mathbb{N}}$. Then the srt $\{e_n : n \in \mathbb{N} \}$ is linearly independent and in particular $e_n \neq 0 \quad \forall n \in \mathbb{N}.$
Proof. Assume this set is linearly dependent. Then there will be \( k_1, k_2, \ldots, k_n \in \mathbb{N} \) and \( \lambda_{k_1}, \lambda_{k_2}, \ldots, \lambda_{k_n} \in \mathbb{R} \), not all zeros, such that \( \lambda_{k_1} e_{k_1} + \lambda_{k_2} e_{k_2} + \ldots + \lambda_{k_n} e_{k_n} n = 0 \). But this is a contradiction by the uniqueness of the expansion for \( 0 \in X \).

It is immediate to show that only a separable Banach space can have a basis. The inverse is not true, as shown by P. Enflo in [3].

**Proposition 1.1.** Let \( X \) be a Banach space with a basis \( \{ e_n \}_{n \in \mathbb{N}} \). Then \( X \) is separable.

**Proof.** Consider the set

\[
D = \left\{ \sum_{i=1}^{n} r_i e_i : \{r_i\}_{i=1}^{n} \subseteq \mathbb{Q}, n \in \mathbb{N} \right\}.
\]

\( D \) is clearly countable. It suffices to show \( D \) is dense in \( X \). Consider \( x = \sum_{n=1}^{\infty} \lambda_n e_n \in X \). Then for any \( \epsilon > 0 \) there is \( N \in \mathbb{N} \) and \( \{r_i\}_{i \in \mathbb{N}} \subseteq \mathbb{Q} \) such that

\[
|| \sum_{i=1}^{n} \lambda_i e_i - x || < \frac{\epsilon}{2^i}, \quad \forall n > N \text{ and } |\lambda_i - r_i||e_i|| < \frac{\epsilon}{2^{i+1}}, \quad \forall i \in \mathbb{N}.
\]

Then for all \( n > N \) we get

\[
|| \sum_{i=1}^{n} r_i e_i - x || \leq \sum_{i=1}^{n} |\lambda_i - r_i||e_i|| + || \sum_{i=1}^{n} \lambda_i e_i - x ||
\]

\[
< \sum_{i=1}^{n} \frac{\epsilon}{2^{i+1}} + \frac{\epsilon}{2} < \epsilon
\]

Hence \( X \) is separable. \( \square \)

We need a concrete criterion to check whether a given sequence is a basis. For this purpose, consider a Banach \( X \) with basis \( \{ e_n \}_{n \in \mathbb{N}} \). For any \( x = \sum_{n=1}^{\infty} \lambda_n e_n \in X \) we clearly get \( \sup_{n \in \mathbb{N}} \left\{ || \sum_{i=1}^{n} \lambda_i e_i || \right\} < \infty \). So we may define

\[
||| x ||| = \sup_{n \in \mathbb{N}} \left\{ || \sum_{i=1}^{n} \lambda_i e_i || \right\}
\]
It is straightforward that $||| \cdot |||$ is a norm in $X$. For any $n \in \mathbb{N}$, let us define the natural projections $P_n : X \to X$, given by

$$P_n(x) = P_n(\sum_{n=1}^{\infty} \lambda_ne_n) = \sum_{i=1}^{n} \lambda_ie_i$$

Clearly the projections are linear operators and the following bounds hold

$$\lim_{n \to \infty} P_n(x) = x \quad \forall x \in X \quad \sup_{n \in \mathbb{N}} \{||P_n(x)||\} = |||x||| \quad \forall x \in X$$

We get the following estimate:

**Proposition 1.1.2.** There is a constant $K > 0$ such that

$$||x|| \leq |||x||| \leq K||x||, \quad \forall x \in X$$

**Proof.** The lower bound is trivially obtained since $||x|| = \lim_{n \to \infty} ||P_n(x)|| \leq |||x|||$. For the upper bound, it suffices to show that $(X, ||\cdot|||)$ is a Banach space. Then the Open Mapping Theorem guarantees the equivalence of $||\cdot||$ and $|||\cdot|||$. For this purpose consider a Cauchy sequence $\{y^{(k)}\}_{k \in \mathbb{N}}$ in $(X, |||\cdot|||)$. Then for any $\epsilon > 0$ there is $k_0 \in \mathbb{N}$ such that

$$|||y^{(i)} - y^{(j)}||| < \epsilon \quad \forall i, j > k_0 \quad \Rightarrow \quad ||P_n(y^{(i)}) - P_n(y^{(j)})|| < \epsilon \quad \forall i, j > k_0, \forall n \in \mathbb{N}$$

The sequence $\{P_n(y^{(k)})\}_{k \in \mathbb{N}}$ is a Cauchy sequence in the finite dimensional space $< e_1, ..., e_n >$. Therefore the sequence $\{P_n(y^{(k)})\}_{k \in \mathbb{N}}$ converges for any $n \in \mathbb{N}$. Let us denote $z_n = \lim_{k \to \infty} P_n(y^{(k)})$. Then $\{z_n\}_{n \in \mathbb{N}}$ is $||\cdot||$-converging. Indeed, consider $\epsilon > 0$ and $k, N \in \mathbb{N}$ such that

$$||P_n(y^{(k)}) - P_n(y^{(j)})|| < \frac{\epsilon}{3} \quad \forall j > k, \forall n \in \mathbb{N}$$

Thus

$$||P_n(y^{(k)}) - z_n|| < \frac{\epsilon}{3} \forall n \in \mathbb{N} \text{ and } ||P_n(y^{(k)}) - P_m(y^{(k)})|| < \frac{\epsilon}{3}, \quad \forall n, m > N$$
Then for any \( n, m > N \), we obtain

\[
\|z_n - z_m\| \leq \|z_n - P_n(y^{(k)})\| + \|P_n(y^{(k)}) - P_m(y^{(k)})\| + \|P_m(y^{(k)}) - z_m\| < \epsilon
\]

hence \( \{z_n\}_{n \in \mathbb{N}} \) is \( \| \cdot \| \)-Cauchy. Since \((X, \| \cdot \|)\) is complete, the sequence \( \{z_n\}_{n \in \mathbb{N}} \) converges. Let

\[
z \equiv \lim_{n \to \infty} z_n.
\]

Then for any \( m > n \in \mathbb{N} \) we have that

\[
P_n(z_m) = P_n(\lim_{k \to \infty} P_m(y^{(k)})) = \lim_{k \to \infty} P_n(P_m(y^{(k)})) = \lim_{k \to \infty} P_n(y^{(k)}) = z_n,
\]

where we used the fact that \( P_n \) is continuous in the finite dimensional space \( <e_1, ..., e_n> \). We claim there is a unique sequence of real numbers \( \{\alpha_i\}_{i \in \mathbb{N}} \) such that \( z_n = \sum_{i=1}^{n} \alpha_i e_i \ \forall n \in \mathbb{N} \). Indeed, \( z_1 \in <e_1> \) so there is unique \( \alpha_1 \in \mathbb{R} \) such that \( z_1 = \alpha_1 e_1 \). Since \( P_1(z_2) = z_1 \) and \( \{e_n\}_{n \in \mathbb{N}} \) is linearly independent, there is unique \( \alpha_2 \in \mathbb{R} \) such that \( z_2 = \alpha_1 e_1 + \alpha_2 e_2 \). Inductively, having defined \( \alpha_1, ... \alpha_n \), we define \( \alpha_{n+1} \) in the same way. Therefore we get \( z = \lim_{n \to \infty} z_n \), so \( P_n(z) = \sum_{i=1}^{n} \alpha_i e_i = z_n \ \forall n \in \mathbb{N} \). Consider \( \epsilon > 0 \). Then there is \( k_0 \in \mathbb{N} \) such that for any \( k > k_0 \), there holds

\[
\sup \left\{ \|z_n - P_n(y^{(k)})\| : n \in \mathbb{N} \right\} < \epsilon
\]

\[
\sup \left\{ \|P_n(z) - P_n(y^{(k)})\| : n \in \mathbb{N} \right\} < \epsilon
\]

\[
\|z - y^{(k)}\| < \epsilon \Rightarrow \lim_{k \to \infty} y^{(k)} = z
\]

Hence \((X, \| \cdot \|)\) is a Banach space and the result follows.

\[\square\]

**Proposition 1.1.3.** For any \( n \in \mathbb{N} \), the natural projection \( P_n \) is a bounded linear operator.

**Proof.** Let \( n \in \mathbb{N} \) and \( x \in X \). Then

\[
\|P_n(x)\| \leq \|x\| \leq K \|x\|
\]

and the claim follows. \[\square\]
Since the projections are continuous and point-wise bounded, the Uniform Bound Principle implies that
\[ \sup_{n \in \mathbb{N}} \| P_n \| < \infty \]
We define \( bc(\{e_n\}_{n \in \mathbb{N}}) = \sup_{n \in \mathbb{N}} \| P_n \| \). and we call this number basis constant.
Then \( bc(\{e_n\}_{n \in \mathbb{N}}) \geq 1 \). Indeed, for any \( x \in X, x \neq 0 \) we have
\[ \| P_n(x) \| \leq \| P_n \| ||x|| \leq bc(\{e_n\}_{n \in \mathbb{N}}) ||x|| \]
so for any \( n \to \infty \), the result follows. If the basis constant equals 1, the basis is called monotone.
For any \( n \in \mathbb{N} \) we \( e_n^* : X \to \mathbb{R} \) by:
\[ e_n^*(x) = e_n^*(\sum_{k=1}^{\infty} \lambda_k e_k) = \lambda_n \]
Clearly \( x = \sum_{n=1}^{\infty} e_n^*(x)e_n \). We immediately get the following

**Proposition 1.1.4.** For any \( n \in \mathbb{N} \), \( \{e_n^*\}_{n \in \mathbb{N}} \) is a bounded linear functional.

**Proof.** Let \( n \in \mathbb{N} \) and \( x = \sum_{n=1}^{\infty} \lambda_n e_n \in X \). Then
\[ |e_n^*(x)| = \frac{||e_n^*(x)e_n||}{||e_n||} = \frac{|| \sum_{i=1}^{n} \lambda_i e_i - \sum_{i=1}^{n-1} \lambda_i e_i ||}{||e_n||} \leq \frac{2||x||}{||e_n||} \leq \frac{2K}{||e_n||} ||x|| \]
which proves the claim. \( \square \)

The functionals \( \{e_n^*\}_{n \in \mathbb{N}} \) are called biorthogonal functionals of the basis \( \{e_n\}_{n \in \mathbb{N}} \). It is clear that both biorthogonal functionals and normal projections are defined with respect to a given basis.

Using all these, we get the following criterion:
Theorem 1.1.1. Let $X$ be a Banach space and $\{e_n\}_{n \in \mathbb{N}} \subseteq X$. The following are equivalent

i) $\{e_n\}_{n \in \mathbb{N}}$ is a Schauder basis of $X$

ii) The following hold:

- $e_n \neq 0$ $\forall n \in \mathbb{N}$
- $[e_n : n \in \mathbb{N}] = X$ where $[e_n : n \in \mathbb{N}] = \langle e_n : n \in \mathbb{N} \rangle$
- $\exists K > 0$ such that for any $m > n \in \mathbb{N}$ and $\lambda_1, ..., \lambda_m \in \mathbb{R}$ we have

$$|| \sum_{i=1}^{n} \lambda_i e_i || \leq K || \sum_{i=1}^{m} \lambda_i e_i ||$$

Proof. i)$\Rightarrow$ii) Let $m > n \in \mathbb{N}$ and $\lambda_1, ..., \lambda_m \in \mathbb{R}$. Then

$$|| \sum_{i=1}^{n} \lambda_i e_i || = || P_n(x) || = || P_n(P_m(x)) || \leq || P_n || || P_m(x) || \leq bc\langle \{e_n\}_{n \in \mathbb{N}} \rangle || \sum_{i=1}^{m} \lambda_i e_i ||$$

ii)$\Rightarrow$i) First notice that the set $\{e_n : n \in \mathbb{N}\}$ is linearly independent. Indeed, without loss of generality, we consider $n \in \mathbb{N}$ and $\lambda_1, ..., \lambda_n \in \mathbb{R}$ such that $\lambda_1 e_1 + ... + \lambda_n e_n = 0$. Then

$$|| \lambda_i || e_i || = || \sum_{j=1}^{i} \lambda_j e_j - \sum_{j=1}^{i-1} \lambda_j e_j || \leq 2K || \sum_{i=1}^{n} \lambda_i e_i || = 0 \Rightarrow \lambda_i = 0 \ \forall i = 1, ..., n$$

For any $n \in \mathbb{N}$, we define

$$p_n : \langle e_n : n \in \mathbb{N} \rangle \rightarrow < e_n : n \in \mathbb{N} > \ \text{with} \ \ p_n(\sum_{i=1}^{m} \lambda_i e_i) = \sum_{i=1}^{\min\{n,m\}} \lambda_i e_i$$

The maps $p_n$ are well defined linear operators since the set $\{e_n : n \in \mathbb{N}\}$ is linearly independent. We also have that $|| p_n(\sum_{i=1}^{m} \lambda_i e_i) || = || \sum_{i=1}^{\min\{n,m\}} \lambda_i e_i || \leq K || \sum_{i=1}^{m} \lambda_i e_i ||$ $p_n$ are also bounded with $|| p_n || \leq K$ $\forall n \in \mathbb{N}$. Since $< e_n : n \in \mathbb{N} >$ is dense $X$, we extend $p_n$ to linear bounded operators $p_n : X \rightarrow < e_n : n \in \mathbb{N} >$ such that $|| p_n || \leq K$ $\forall n \in \mathbb{N}$. It is straightforward that for all $m > n$
and $x \in X$ we get $p_n(x) = p_n(p_m(x))$. Therefore, with a argument similar to Proposition 1.1.2 we get that for any $x \in X$ there is unique sequence $\{\lambda_i\}_{i \in \mathbb{N}}$ such that $p_n(x) = \sum_{i=1}^{n} \lambda_i e_i \quad \forall n \in \mathbb{N}$. It remains to show that $\lim_{n \to \infty} p_n(x) = x$.

Indeed, let $\epsilon > 0$. Then there is $z = \sum_{i=1}^{k} \beta_i e_i \in \epsilon e_n : n \in \mathbb{N}$ with $||x - z|| < \frac{\epsilon}{K + 1}$. Then, for $m > k$ we have $p_m(z) = z$. So

$$||x - p_m(x)|| \leq ||x - z|| + ||z - p_m(z)|| + ||p_m(z) - p_m(x)||$$

$$\leq (1 + ||p_m||)||x - z|| < \epsilon$$

and the result follows. \(\square\)

**Example 1.1.1.** Let $1 < p < \infty$. Then the sequence $e_n = (0, 0, ..., 1, 0, ...)$ (n-position) is a monotone basis of $\ell_p$. We call it the natural basis of $\ell_p$. $c_0$ has the same basis as well.

**Proof.** Let $x = (\lambda_1, \lambda_2, ..., \lambda_n, ...) \in \ell_p$. Define $s_n = \sum_{i=1}^{n} \lambda_i e_i$. Then we get

$$||s_n - x|| = (\sum_{i=n+1}^{\infty} |\lambda_i|^p)^{1/p} \xrightarrow{n \to \infty} 0$$

Moreover, for all $m > n \in \mathbb{N}$ and $\lambda_1, ..., \lambda_m \in \mathbb{R}$ we have

$$||\sum_{i=1}^{n} \lambda_i e_i|| = (\sum_{i=1}^{n} ||\lambda_i|^p)^{1/p} \leq (\sum_{i=1}^{m} ||\lambda_i|^p)^{1/p} = (\sum_{i=1}^{m} \lambda_i e_i)||$$

and the claim follows. We similarly can see that $c_0$ has the same basis. \(\square\)

**Remark 1.1.2.** Let $X$ be a Banach space with basis $\{e_n\}_{n \in \mathbb{N}}$ and consider the set of real numbers $K \geq 0$ such that for all $m > n \in \mathbb{N}$ and $\lambda_1, ..., \lambda_m \in \mathbb{R}$ there holds

$$||\sum_{i=1}^{n} \lambda_i e_i|| \leq K||\sum_{i=1}^{m} \lambda_i e_i||$$

Then the constant $o$ the basis is the minimum of this set.
Proof. It is shown in Theorem (1.1.7) that the basis constant $bc(\{e_n\}_{n \in \mathbb{N}})$ has this property. It suffices to show it is the smallest such number. Indeed, consider $K$ in this set. Then for all $x \in X$ with $||x|| \leq 1$ and $n \in \mathbb{N}$ we get $||P_n(x)|| \leq K||P_m(x)|| \ \forall m > n$. Therefore letting $n \to \infty$ we get $||P_n(x)|| \leq K||x|| \leq K$ and the result is proved. \qed

Before the end of this section, we present a useful criterion for the point-wise convergence of linear and bounded functionals.

Proposition 1.1.5. Let $X$ be a Banach space. We consider $x^* \in X^*, x^* \neq 0$ and $\{x^*_n\}_{n \in \mathbb{N}} \subseteq X^*$ such that $\sup_{n \in \mathbb{N}} ||x^*_n|| = M < \infty$.

If there is norm-dense $S \subseteq X$ such that $x^*_n(s) \xrightarrow{k \to \infty} x^*(s) \ \forall s \in S$, then $x^* = w^* - \lim_{n \to \infty} x^*_n$.

Proof. We will show that $x^*(x) = \lim_{n \to \infty} x^*_n(x) \ \forall x \in X$. Consider $x \in X$. By density, there is $\{s_n\}_{n \in \mathbb{N}} \subseteq S$ with $x = \lim_{n \to \infty} s_n$. Let $\epsilon > 0$ and consider $N, n_0 \in \mathbb{N}$ such that

$$||s_N - x|| < \frac{\epsilon}{3 \max \{M, ||x^*||\}} \quad \text{and} \quad |x^*_n(s_N) - x^*(s_N)| < \frac{\epsilon}{3} \ \forall n \geq n_0.$$

Then for any $n \geq n_0$, we have

$$|x^*_n(x) - x^*(x)| \leq |x^*_n(x) - x^*_n(s_N)| + |x^*_n(s_N) - x^*(s_N)| + |x^*(s_N) - x^*(x)| \leq M||s_N - x|| + \frac{\epsilon}{3} + ||x^*||||s_N - x|| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

The result is proved. \qed

Corollary 1.1.1. Let $X$ be a Banach space with basis $\{e_n\}_{n \in \mathbb{N}}$. We consider $x^* \in X^*, x^* \neq 0$ and $\{x^*_k\}_{k \in \mathbb{N}}$ a sequence in $X^*$ with $\sup_{k \in \mathbb{N}} ||x^*_k|| < \infty$.

Then if $x^*_k(e_n) \xrightarrow{k \to \infty} x^*(e_n) \ \forall n \in \mathbb{N}$, we obtain $x^* = w^* - \lim_{k \to \infty} x^*_k$.

Proof. Defining $S = \langle e_n : n \in \mathbb{N} \rangle$ and using linearity of $\{x^*_k\}_{k \in \mathbb{N}}$ and $x^*$ we obtain the result using last Proposition. \qed
1.2 Basic sequences

Definition 1.2.1. Let $X$ be a Banach space and $\{x_n\}_{n \in \mathbb{N}}$ a sequence of distinct, non-zero elements of $X$. The sequence $\{x_n\}_{n \in \mathbb{N}}$ is called basic sequence if it is Schauder basis of the subspace $[x_n : n \in \mathbb{N}]$. We define the constant of the basic sequence $\{x_n\}_{n \in \mathbb{N}}$ as the constant of the Schauder basis of $[x_n : n \in \mathbb{N}]$. Finally, we define the biorthogonal functionals of $\{x_n\}_{n \in \mathbb{N}}$ as $x^*_n : [x_n : n \in \mathbb{N}] \to \mathbb{R}$ with $x^*_n(\sum_{k=1}^{\infty} \alpha_k x_k) = \alpha_n$.

We immediately get the following characterization for basic sequences:

Proposition 1.2.1. Let $X$ be a Banach space and consider the sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$. The following are equivalent:

i) $\{x_n\}_{n \in \mathbb{N}}$ is basic

ii) The following hold:

- $x_n \neq 0 \ \forall n \in \mathbb{N}$

  - $\exists K > 0$ such that for all $m > n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ there holds

$$||\sum_{i=1}^{n} \lambda_i x_i|| \leq K ||\sum_{i=1}^{m} \lambda_i x_i||.$$

Remark 1.2.1. Let $X$ be a Banach space with basis $\{e_n\}_{n \in \mathbb{N}}$ and constant $K$. Then the biorthogonal functionals form a basic sequence of $X^*$. Moreover, for any $x^* \in [e^*_n : n \in \mathbb{N}]$ the following expansion holds:

$$x^* = \sum_{n=1}^{\infty} x^*(e_n)e^*_n.$$

Proof. Let $m > n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. Consider $x \in X$ with $||x|| \leq 1$. Then

$$\left| \sum_{i=1}^{n} \lambda_i e^*_i(x) \right| = \left| \sum_{i=1}^{m} \lambda_i e^*_i(P_n(x)) \right| = ||P_n^*|| \left| \sum_{i=1}^{m} \lambda_i e^*_i \right| \leq K ||\sum_{i=1}^{m} \lambda_i e^*_i||$$

hence $\{e^*_n\}_{n \in \mathbb{N}}$ is basic. Therefore, for any $x^* \in [e^*_n : n \in \mathbb{N}]$ there is a unique sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ such that $x^* = \sum_{n=1}^{\infty} \lambda_n e^*_n$, thus $x^*(e_n) = \alpha_n \ \forall n \in \mathbb{N}$. 

\[\square\]
Notice that this result does not imply that the biorthogonal functionals of a Schauder basis form a basis of the dual space. We will now show that every Banach space contains a basic sequence. We will use the following Lemma, due to Mazur.

**Lemma 1.2.1.** (S. Mazur) Let $X$ be a Banach space and $F$ be a finite dimensional subspace. Then for any $\epsilon > 0$, there is $x \in X$ with $||x|| = 1$ such that

$$||y|| \leq (1 + \epsilon)||y + mx|| \quad \forall y \in F \quad \forall m \in \mathbb{R}.$$ 

**Proof.** Since $F$ is finite dimensional, the unit sphere $S_F$ is $|| \cdot ||$-compact so there is $k \in \mathbb{N}$ and $y_1, ..., y_k \in S_F$ such that $S_F \subseteq \bigcup_{i=1}^{k} B(y_i, \frac{\epsilon}{2})$. We denote $B(y, \epsilon)$ the open ball of center $y$ and radius $\epsilon$. Hahn-Banach Theorem implies that for any $j \in \{1, ..., k\}$ we may find $y^*_i \in X^*$ with $y^*_i(y_i) = ||y_i|| = 1$. The subspace $\bigcap_{i=1}^{k} \ker(y^*_i)$ has finite co-dimension, since all kernels are of 1-co-dimension. Since $X$ is infinite dimensional, there is $x \in X$ with $||x|| = 1$ and $y^*_i(x) = 0 \quad \forall i = 1, ..., k$. Consider $y \in S_F$ and $0 < \epsilon < 1$. Then there is $j \in \{1, ..., k\}$ such that $||y - y_j|| < \frac{\epsilon}{2}$. Therefore, for any $m \in \mathbb{R}$ we get

$$||y + mx|| \geq ||y_j + mx|| - ||y - y_j|| \geq y^*_j(y_j + mx) - \frac{\epsilon}{2} = y^*_j(y_j) - \frac{\epsilon}{2} = ||y_j|| - \frac{\epsilon}{2}$$

$$= 1 - \frac{\epsilon}{2} \geq \frac{1}{1 + \epsilon}, \quad \text{since } 0 < \epsilon < 1$$

Therefore, for any $\epsilon > 0$, $y \in B_F$ and $m \in \mathbb{R}$ there is $x \in B_X$ with $(1 + \epsilon)||y + mx|| \geq 1$. We consider $y \in F$ with $y \neq 0$. Then for all $\epsilon > 0$ and $m \in \mathbb{R}$, there is $x \in S_X$ such that

$$1 \leq (1 + \epsilon) \left| \frac{y}{||y||} + \frac{mx}{||y||} \right| \Rightarrow ||y|| \leq (1 + \epsilon)||y + mx||$$

The result is proved. \qed

**Theorem 1.2.1.** (S. Banach) Every Banach space contains a basic sequence. Equivalently, every Banach space contains a closed subspace with Schauder basis.
Proof. Consider $\epsilon > 0$. By Intermediate Value Theorem, we may consider a sequence of positive numbers $\{\epsilon_n\}_{n \in \mathbb{N}}$ such that $\ln(1+\epsilon_n) \leq \frac{\ln(1+\epsilon)}{2^n} \quad \forall n \in \mathbb{N}$. So, we get

$$\sum_{i=1}^{n} \ln(1+\epsilon_n) \leq \ln(1+\epsilon) \quad \forall n \in \mathbb{N} \Rightarrow \prod_{n=1}^{\infty} (1+\epsilon_n) \leq 1 + \epsilon$$

We construct the required sequence inductively. Let $x_1 \in X$ with $||x_1|| = 1$. We define $F_1 = \{x_1\}$. Mazur’s Lemma implies that there is $x_2 \in X$ with $||x_2|| = 1$ such that $||y|| \leq (1 + \epsilon_2)||y + mx_2|| \quad \forall y \in F_1, \forall m \in \mathbb{R}$

Define $F_2 = \{x_1, x_2\}$. Mazur’s Lemma implies that there is $x_3 \in X$ with $||x_3|| = 1$ such that $||y|| \leq (1 + \epsilon_3)||y + mx_3|| \quad \forall y \in F_2, \forall m \in \mathbb{R}$.

Inductively, we define $F_n = \{x_1, x_2, \ldots, x_n\}$ and pick $x_{n+1} \in X$ with $||x_{n+1}|| = 1$ such that $||y|| \leq (1 + \epsilon_{n+1})||y + mx_{n+1}|| \quad \forall y \in F_n, \forall m \in \mathbb{R}$.

Hence, we have defined a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$. We will show this sequence is basic of constant smaller than $1 + \epsilon$. Indeed, let $m > n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$. Define

$$y_k = \sum_{i=1}^{k} \alpha_i e_i \in F_k \quad \forall k \in \mathbb{N}$$

Then

$$|| \sum_{i=1}^{n} \alpha_i e_i || = ||y_n|| \leq (1 + \epsilon_{n+1})||y_n + \alpha_{n+1}x_{n+1}||$$

$$= (1 + \epsilon_{n+1})||y_{n+1}||$$

$$\leq (1 + \epsilon_{n+1})(1 + \epsilon_{n+2})||y_{n+1} + \alpha_{n+2}x_{n+2}||$$

$$= (1 + \epsilon_{n+1})(1 + \epsilon_{n+2})||y_{n+3}||$$

$$\leq \prod_{i=n+1}^{m} (1 + \epsilon_i)||y_m|| \leq (1 + \epsilon)|| \sum_{i=1}^{m} \alpha_i e_i ||$$

Thus, the sequence $\{x_n\}_{n \in \mathbb{N}}$ is basic of constant smaller than $1 + \epsilon$. \qed

1.3 Equivalence of basic sequences

Definition 1.3.1. Let $X, Y$ be Banach spaces and $\{x_n\}_{n \in \mathbb{N}} \subseteq X, \{y_n\}_{n \in \mathbb{N}} \subseteq Y$ be basic sequences. The sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ are called equiv-
alent if there exist $c, C > 0$ such that for any $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ there holds
\[
c \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \leq \left\| \sum_{i=1}^{n} \alpha_i y_i \right\| \leq C \left\| \sum_{i=1}^{n} \alpha_i x_i \right\|
\]

Proposition 1.3.1. Let $X, Y$ be Banach spaces and $\{x_n\}_{n \in \mathbb{N}} \subseteq X, \{y_n\}_{n \in \mathbb{N}} \subseteq Y$ be basic sequences. Then the following are equivalent:

i) $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ are equivalent.

ii) If $\{\alpha_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is a sequence of numbers such that $\sum_{n=1}^{\infty} \alpha_n x_n$ converges then, $\sum_{n=1}^{\infty} \alpha_n y_n$ converges and vice versa.

iii) There is an isomorphism $T : [x_n : n \in \mathbb{N}] \to [y_n : n \in \mathbb{N}]$ with $T(x_n) = y_n \quad \forall n \in \mathbb{N}$.

Proof. i) $\Rightarrow$ ii) Let $\{\alpha_n\}_{n \in \mathbb{N}}$ such that $\sum_{n=1}^{\infty} \alpha_n x_n$ converges. We will show the sequence $\left\{ \sum_{i=1}^{n} \alpha_i y_i \right\}_{n \in \mathbb{N}}$ is Cauchy. Indeed, for any $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $m > n > N$, we have
\[
\left\| \sum_{i=n+1}^{m} \alpha_i x_i \right\| < \frac{\epsilon}{C}
\]
Then we get
\[
\left\| \sum_{i=n+1}^{m} \alpha_i y_i \right\| < \epsilon
\]
The other way is identical.

ii) $\Rightarrow$ iii) We define $T : [x_n : n \in \mathbb{N}] \to Y$ with $T(x) = T\left( \sum_{n=1}^{\infty} \alpha_n x_n \right) = \sum_{n=1}^{\infty} \alpha_n y_n$. The map $T$ is well defined and linear because the sequence $\{x_n\}_{n \in \mathbb{N}}$
is basic and we have assumed \( ii) \) holds. For any \( n \in \mathbb{N} \) we define

\[
T_k : X \to \langle y_1, \ldots, y_n \rangle \quad \text{with} \quad T_n(\sum_{i=1}^{\infty} \alpha_i x_i) = \sum_{i=1}^{n} \alpha_i y_i
\]

and

\[
F_n : \langle x_1, \ldots, x_n \rangle \to \langle y_1, \ldots, y_n \rangle \quad \text{with} \quad F_n(\sum_{i=1}^{n} \alpha_i x_i) = \sum_{i=1}^{n} \alpha_i y_i
\]

Then for any \( n \in \mathbb{N} \), we take \( T_n = F_n \circ P_n \) where \( P_n \) stand for the projections of the basic sequence \((y_n)_{n \in \mathbb{N}}\). It is clear \( T_n \) is linear and continuous, since \( F_n \) linear operator on a finite dimensional space. Therefore, Banach-Steinhauss Theorem implies \( T \) is bounded since \( T(x) = \lim_{n \to \infty} T_n(x) \forall x \in X \)

Moreover \( T \) is a bijection since for \( y = \sum_{n=1}^{\infty} \alpha_n y_n \) we define \( x = \sum_{n=1}^{\infty} \alpha_n x_n \) and get \( T(x) = y \). The Open Mapping theorem implies \( T \) is an isomorphism.

\[ \text{iii) } \Rightarrow \text{ i) Comes clearly for } c = \frac{1}{||T^{-1}||} \text{ and } C = ||T||. \]

\[ \square \]

**Remark 1.3.1.** Let \( X \) be a Banach space with basis \( \{x_n\}_{n \in \mathbb{N}} \) and constant \( K \) and \( Y \) a Banach space. If there is a sequence \( \{y_n\}_{n \in \mathbb{N}} \) in \( Y \) and \( c, C > 0 \) such that for any \( n \in \mathbb{N} \) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) there holds

\[ c \| \sum_{i=1}^{n} \alpha_n x_n \| \leq \| \sum_{i=1}^{n} \alpha_n y_n \| \leq C \| \sum_{i=1}^{n} \alpha_n x_n \| \]

Then \( X \) embeds isomorphically in \( Y \).

**Proof.** By Proposition 1.3.1, it is enough to show that the sequence \( \{y_n\}_{n \in \mathbb{N}} \) is basic. Indeed, let \( m > n \in \mathbb{N} \) and \( \alpha_1, \ldots, \alpha_m \in \mathbb{R} \). Then

\[ \| \sum_{i=1}^{n} \alpha_i y_i \| \leq C \| \sum_{i=1}^{m} \alpha_i x_i \| \leq \frac{CK}{c} \| \sum_{i=1}^{m} \alpha_i y_i \| \]

and the claim is proved \[ \square \]

We now present a way to construct equivalent sequences. Before that we need a related Lemma.
Lemma 1.3.1. Let $X$ be a Banach space and $T : X \to X$ a linear operator. If there is $\delta \in (0, 1)$ such that for all $x \in X$, there holds $\|x - T(x)\| \leq \delta \|x\|$, then $T$ is an isomorphism.

**Proof.** We may see directly see $T$ is an isomorphic embedding. Indeed,

\[
\|T(x)\| - \|x\| \leq \delta \|x\| \Rightarrow \|T(x)\| \leq (1 + \delta)\|x\|
\]

\[
\|x\| - \|T(x)\| \leq \delta \|x\| \Rightarrow (1 - \delta)\|x\| \leq \|T(x)\|
\]

In order to show $T$ is surjective, we argue by contradiction. Assume that $T(X)$ is a strict subspace of $X$. Since $T$ is an embedding, $T(X)$ is complete and therefore closed subspace. Therefore, by Hahn-Banach Theorem, we may find $x^*_0 \in S_{X^*}$ with $x^*_0(x) = 0$, $\forall x \in T(X)$. Since $0 < \delta < 1$, we pick $x_0 \in X$ with $\|x_0\| = 1$ and $x^*_0(x_0) > \delta$. Thus we obtain

\[
\|x_0 - T(x_0)\| = \sup \{x^*(x_0 - T(x_0)) : \|x^*\| = 1\} \Rightarrow \|x_0 - T(x_0)\| \geq x^*_0(x_0 - T(x_0)) = x^*_0(x_0) > \delta = \delta\|x_0\|
\]

which is a contradiction. \hfill $\square$

We conclude the following:

**Proposition 1.3.2. (Small Perturbation Lemma)** Let $X$ be a Banach space and $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}$ sequences in $X$ such that $\{x_n\}_{n \in \mathbb{N}}$ is basic with constant $K$. Let $\delta = \inf \{\|x_n\| : n \in \mathbb{N}\} > 0$. If

\[
\sum_{n=i}^{\infty} \|x_n - y_n\| < \frac{\delta}{3K}
\]

then the sequence $\{y_n\}_{n \in \mathbb{N}}$ is basic and equivalent to $\{x_n\}_{n \in \mathbb{N}}$.

**Proof.** It suffices to define an isomorphism $T : X \to X$ with $T(x_n) = y_n$. Indeed for any $m > n$ and $\alpha_1, ..., \alpha_m \in \mathbb{R}$, we have

\[
\|\sum_{i=1}^{n} \alpha_i y_i\| \leq \|T\|\|\sum_{i=1}^{n} \alpha_i x_i\| \leq \|T\|K\|\sum_{i=1}^{m} \alpha_i x_i\| \leq \|T\|K\|T^{-1}\|\sum_{i=1}^{m} \alpha_i y_i\|
\]

Therefore the sequence $\{y_n\}_{n \in \mathbb{N}}$ is basic. It is also equivalent to $\{x_n\}_{n \in \mathbb{N}}$ since for all $n \in \mathbb{N}$ and $\alpha_1, ..., \alpha_n \in \mathbb{R}$, we have

\[
\frac{1}{\|T^{-1}\|}\|\sum_{i=1}^{n} \alpha_i x_i\| \leq \|\sum_{i=1}^{n} \alpha_i y_i\| \leq \|T\|\|\sum_{i=1}^{n} \alpha_i x_i\|
\]
Let $n \in \mathbb{N}$. Then for any $x \in [x_n : n \in \mathbb{N}]$ we get
\[
|x_n^*(x)||x_n| = \left\| \sum_{i=1}^{n} x_i^*(x)e_i - \sum_{i=1}^{n-1} x_i^*(x)e_i \right\||x_n| \leq 2K||x|| \Rightarrow ||x_n^*|| \leq \frac{2K}{\delta}
\]
For $n \in \mathbb{N}$ we use Hahn-Banach Theorem to extend $x_n^*$ in $X^*$ with conservation of the norm. Then for any $x \in X$, the series $\sum_{n=1}^{\infty} x_n^*(x)(y_n - x_n)$ is absolutelt convergent and thus converges since $X$ is a Banach space. Indeed,
\[
\sum_{n=1}^{\infty} |x_n^*(x)||y_n - x_n| \leq \frac{2K}{\delta}||x|| \sum_{n=1}^{\infty} |y_n - x_n| < \frac{2}{3}||x||
\]
Let us define
\[
T : X \rightarrow X \text{ by } T(x) = x + \sum_{n=1}^{\infty} x_n^*(x)(y_n - x_n)
\]
Then $T$ is linear, we have that $T(x_n) = y_n$ $\forall n \in \mathbb{N}$ and
\[
||x - T(x)|| = \left\| \sum_{n=1}^{\infty} x_n^*(x)(y_n - x_n) \right\| \leq ||x|| \sum_{n=1}^{\infty} ||x_n^*|||y_n - x_n| < \frac{2}{3}||x||.
\]
By Lemma 1.3.1, $T$ is an isomorphism. \hfill \Box

We mention, without proof, a very important Theorem which will turn out to be very important. As known, $\ell_1$ cannot embed in a space with separable dual since its dual coincides with $\ell_\infty$. H. Rosenthal proved a partial inverse of this, the famous $\ell_1$-Theorem.

**Theorem 1.3.1.** (\(\ell_1\)-Theorem) Let $X$ be a Banach space and \(\{x_n\}_{n \in \mathbb{N}}\) a bounded sequence in $X$. Then there holds exclusively one of the following:

i) The sequence \(\{x_n\}_{n \in \mathbb{N}}\) has weak-Cauchy subsequence.

ii) The sequence \(\{x_n\}_{n \in \mathbb{N}}\) is basic and equivalent to the standard basis of $\ell_1$.

The one direction is immediate since the standard basis of $\ell_1$ cannot have a weak-Cauchy subsequence. The other direction is much more complicated though. Reader can find more in [7]. As an application of Rosenthal’s Theorem we prove the following
Proposition 1.3.3. Let $X$ a Banach space not containing $\ell_1$. Then every infinite dimensional subspace of $X$ has a weak-Cauchy unitary sequence.

Proof. Let $Y$ be infinite dimensional subspace of $X$. Then $B_Y$ is not norm-compact. Therefore there is $\{s_n\}_{n\in\mathbb{N}} \subseteq B_Y$ with non-convergent subsequence. But $\ell_1$-Theorem and the assumption that $\ell_1$ does not embed in $X$ the sequence $\{s_n\}_{n\in\mathbb{N}}$ has a subsequence $\{s_{n_k}\}_{k\in\mathbb{N}}$ which is weak-Cauchy. So the sequence $\{s_{n+1} - s_n\}_{k\in\mathbb{N}}$ is weak null. Moreover, since $\{s_{n_k}\}_{k\in\mathbb{N}}$ is not norm-convergent, there is $\theta > 0$ such that the following holds:

$$\forall n \in \mathbb{N} \quad \exists k, m \in \mathbb{N} : n < k < m \quad \text{and} \quad \|s_k - s_m\| \geq \theta \quad (1.1)$$

By (1.1) we determine an increasing sequence $\{p_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$\|s_{p_{2n}} - s_{p_{2n-1}}\| \geq \theta \quad \forall n \in \mathbb{N}$$

Defining $u_n = s_{p_{2n}} - s_{p_{2n-1}}$ we get $u_n \xrightarrow{w} 0$ and $\|u_n\| \geq \theta \quad \forall n \in \mathbb{N}$. Hence, the sequence $\{z_n\}_{n\in\mathbb{N}}$ defined by $z_n = \frac{u_n}{\|u_n\|}$ is unitary and weak null. \hfill $\square$

1.4 Block sequences

In this section we introduce a useful tool, the block sequences.

Definition 1.4.1. Let $X$ be a Banach space with basis $\{e_n\}_{n\in\mathbb{N}}$. A sequence $\{u_n\}_{n\in\mathbb{N}}$ of non-zero elements of $X$ will be called block sequence if there is a sequence of real numbers $\{\alpha_i\}_{i\in\mathbb{N}}$ and an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of positive integers such that for any $k \in \mathbb{N}$, there holds

$$u_k = \sum_{i=n_k+1}^{n_{k+1}} \alpha_i e_i$$

Proposition 1.4.1. Let $X$ be a Banach space with basis $\{e_n\}_{n\in\mathbb{N}}$ and constant $K$. Then every block sequence is basic of constant less or equal than $K$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ a block sequence. Then $u_n \neq 0$ $\forall n \in \mathbb{N}$ and for all $m > k \in \mathbb{N}$ and $\lambda_1, ..., \lambda_m \in \mathbb{R}$ we have
\[
\sum_{j=1}^{k} \lambda_j u_j = \sum_{i=n_j+1}^{n_{j+1}} \alpha_i e_i = \sum_{j=1}^{k} \sum_{i=n_j+1}^{n_{j+1}} \lambda_j \alpha_i e_i \\
\leq K \sum_{j=1}^{m} \sum_{i=n_j+1}^{n_{j+1}} \lambda_j \alpha_i e_i \leq K \sum_{j=1}^{m} \lambda_j u_j
\]

so the sequence \( \{u_n\}_{n \in \mathbb{N}} \) is basic with constant less or equal than \( K \). \( \square \)

**Lemma 1.4.1.** (Sliding hump argument I) Let \( X \) be a Banach space with basis \( \{e_n\}_{n \in \mathbb{N}} \) and a sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( X \) such that \( \delta = \inf_{n \in \mathbb{N}} ||x_n|| > 0 \) and \( \lim_{n \to \infty} e^*_k(x_n) = 0 \) \( \forall k \in \mathbb{N} \). Then for any \( \epsilon > 0 \) there is a subsequence \( \{x'_n\}_{n \in \mathbb{N}} \) and a block \( \{u_n\}_{n \in \mathbb{N}} \) of \( \{e_n\}_{n \in \mathbb{N}} \) such that \( \sum_{n=1}^{\infty} ||x'_n - u_n||^2 < \epsilon^2 \). Moreover, if \( ||x|| = 1 \) \( \forall n \in \mathbb{N} \) there is a further subsequence \( \{x''_n\}_{n \in \mathbb{N}} \) and unitary block \( \{b_n\}_{n \in \mathbb{N}} \) of \( \{e_n\}_{n \in \mathbb{N}} \) such that \( \sum_{n=1}^{\infty} ||x''_n - b_n||^2 < \epsilon^2 \).

**Proof.** We first show that \( \lim_{n \to \infty} P_k(y_n) = 0 \) \( \forall k \in \mathbb{N} \). Indeed, let \( k \in \mathbb{N} \) and \( \epsilon > 0 \). We consider \( N \in \mathbb{N} \) such that for all \( i = 1, \ldots, k \), we have

\[
|e^*_i(x_n)||e_i| < \frac{\epsilon}{2^i} \quad \forall n > N
\]

Then for any \( n > N \)

\[
||P_k(x_n)|| = || \sum_{i=1}^{k} e^*_i(x_n)e_i || \leq \sum_{i=1}^{k} |e^*_i(x_n)||e_i| < \epsilon \quad (1.2)
\]

Consider \( 0 < \epsilon < \delta \). Using (1.2) we will construct the required sequences.

Let \( k_1 = 1 \) and \( n_1 = 0 \). Then \( x_{k_1} = \sum_{i=1}^{\infty} \alpha^{(1)}_i e_i \), so there is \( n_2 \in \mathbb{N} \) with \( n_2 > n_1 \) and

\[
|| \sum_{n=n_2+1}^{\infty} \alpha^{(1)}_n e_n || < \frac{\epsilon}{\sqrt{2}} \Rightarrow
\]

\[
|| \sum_{n=1}^{\infty} \alpha^{(1)}_n e_n - \sum_{n=1}^{n_2} \alpha^{(1)}_n e_n || < \frac{\epsilon}{\sqrt{2}}
\]

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Defining \( u_1 = \sum_{n=n_1+1}^{n_2} \alpha^{(1)}_n e_n \), we get \( ||x_{k_1} - u_1||^2 < \frac{\epsilon^2}{2} \). So by (1.2) there is \( k_2 > k_1 \) such that for

\[
x_{k_2} = \sum_{i=1}^{\infty} \alpha^{(2)}_i e_i
\]

we get

\[
|| \sum_{i=1}^{n_2} \alpha^{(2)}_i e_i || < \frac{\epsilon}{4}.
\]

We consider \( n_3 > n_2 \) such that

\[
|| \sum_{i=n_3+1}^{\infty} \alpha^{(2)}_i e_i || < \frac{\epsilon}{4}.
\]

Equivalently

\[
|| \sum_{i=1}^{\infty} \alpha^{(2)}_i e_i - \sum_{i=1}^{n_2} \alpha^{(2)}_i e_i - \sum_{i=n_2+1}^{n_3} \alpha_i e_i || < \frac{\epsilon}{4}
\]

Setting \( u_2 = \sum_{i=n_2+1}^{n_3} \alpha^{(2)}_i e_i \), we have \( ||x_{k_2} - u_2|| < \frac{\epsilon}{2} \Rightarrow ||x_{k_2} - u_2||^2 < \frac{\epsilon^2}{4} \)

Proceeding inductively, we may find a sequence of positive integers \( \{k_n\}_{n \in \mathbb{N}} \) and a block \( \{u_n\}_{n \in \mathbb{N}} \) of \( \{e_n\}_{n \in \mathbb{N}} \) such that

\[
1 - \epsilon < 1 - m < ||u_n|| \quad \forall n \in \mathbb{N} \quad \text{and} \quad \sum_{n=1}^{\infty} ||x'_{n} - u_n||^2 < m^2 = \frac{(1-\epsilon)^2 \epsilon^2}{4}
\]

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Let us define $z_n = \frac{u_n}{||u_n||} \Rightarrow ||z_n|| = 1 \ \forall n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$, we have

$$||x'_n - z_n|| \leq \frac{||x'_n||u_n|| - u_n||}{||u_n||} \leq \frac{||x'_n||u_n|| - u_n||}{1 - \epsilon} \leq \frac{||x'_n||u_n|| - 1 - u_n||}{1 - \epsilon}$$

$$= \frac{||u_n|| - 1}{1 - \epsilon} + \frac{||x'_n - u_n||}{1 - \epsilon}$$

$$\Rightarrow ||x'_n - z_n||^2 \leq 2 \left(\frac{||u_n|| - 1}{1 - \epsilon}\right)^2 + 2 \frac{||x'_n - u_n||^2}{(1 - \epsilon)^2}$$

Since $\lim_{n \to \infty} ||x'_n - u_n|| = 0$, we have $\lim_{n \to \infty} ||u_n|| = 1$, so we may find a subsequence $\{u'_n\}_{n \in \mathbb{N}}$ of $\{u_n\}_{n \in \mathbb{N}}$ with $\sum_{n=1}^{\infty} \left| ||u'_n|| - 1 \right|^2 < \frac{(1 - \epsilon)^2 \epsilon^2}{4}$. Denoting by $\{x''_n\}_{n \in \mathbb{N}}$ the corresponding subsequence of $\{x'_n\}_{n \in \mathbb{N}}$ and by $\{b_n\}_{n \in \mathbb{N}}$ the corresponding subsequence of $\{z_n\}_{n \in \mathbb{N}}$, we obtain

$$\sum_{n=1}^{\infty} ||x''_n - b_n|| \leq \frac{2}{(1 - \epsilon)^2} \sum_{n=1}^{\infty} \left| ||u'_n|| - 1 \right|^2 + \frac{2}{(1 - \epsilon)^2} \sum_{n=1}^{\infty} ||x''_n - u'_n||^2 < \epsilon^2$$

and the result is proved. \qed

**Lemma 1.4.2. (Sliding Hump Argument II)** Let $X$ be a Banach space with basis $\{e_n\}_{n \in \mathbb{N}}$ and a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$ such that $\delta = \inf_{n \in \mathbb{N}} ||x_n|| > 0$ and $\lim_{n \to \infty} e^*_k(x_n) = 0 \ \forall k \in \mathbb{N}$. Then for any $\epsilon > 0$ there is subsequence of $\{x'_n\}_{n \in \mathbb{N}}$ and a block $\{u_n\}_{n \in \mathbb{N}}$ of $\{e_n\}_{n \in \mathbb{N}}$ such that $\sum_{n=1}^{\infty} ||x'_n - u_n|| < \epsilon$. In particular, for $\epsilon = \frac{\delta}{3K}$, where $K$ is the constant of $\{e_n\}_{n \in \mathbb{N}}$, we may find a subsequence $\{x'_n\}_{n \in \mathbb{N}}$ and block $\{u_n\}_{n \in \mathbb{N}}$ which are equivalent.
Proof. Similarly Lemma 1.4.1, we construct the required sequences. For \( \varepsilon = \frac{\delta}{3K} \), those two sequences are equivalent due to Small Perturbation Lemma I and the fact that \( \{x_n\}_{n \in \mathbb{N}} \) is basic. \qed

**Proposition 1.4.2.** Let \( X \) be a Banach space with basis \( \{x_n\}_{n \in \mathbb{N}} \) which contains isomorphically \( \ell_1 \). Then there is a block of \( \{x_n\}_{n \in \mathbb{N}} \) equivalent to the standard basis of \( \ell_1 \).

**Proof.** Since \( \ell_1 \) embedds in \( X \), there is a sequence \( \{y_n\}_{n \in \mathbb{N}} \) in \( X \) which is basic and equivalent to the standard basis of \( \ell_1 \). Therefore, there are \( m, M > 0 \) such that \( m \leq ||y_n|| \leq M \) \( \forall n \in \mathbb{N} \). For any \( k \in \mathbb{N} \), we have \( ||x_k^*(y_n)|| \leq ||x_k^*||M \) \( \forall n \in \mathbb{N} \) so \( \{x_k^*(y_n)\}_{n \in \mathbb{N}} \) is bounded for all \( k \in \mathbb{N} \). Therefore, with a diagonal argument, we may construct a subsequence \( \{y'_n\}_{n \in \mathbb{N}} \) of \( \{y_n\}_{n \in \mathbb{N}} \) such that the sequence \( \{x_k^*(y'_n)\}_{n \in \mathbb{N}} \) converges for all \( k \in \mathbb{N} \). Let us denote \( z_n = y'_{n+1} - y'_n \). For any \( k \in \mathbb{N} \), we have \( \lim_{n \to \infty} x_k^*(z_n) = 0 \). SO there are \( c, C > 0 \) such that for any \( m \in \mathbb{N} \) and \( \alpha_1, ..., \alpha_m \in \mathbb{R} \) there holds

\[
\begin{align*}
c \sum_{i=1}^{m} |\alpha_i| &\leq || \sum_{i=1}^{m} \alpha_i z_i || \leq C \sum_{i=1}^{m} |\alpha_i| \Rightarrow \inf_{n \in \mathbb{N}} \{||z_n||\} > 0
\end{align*}
\]

Clearly the sequence \( \{z_n\}_{n \in \mathbb{N}} \) is basic. Therefore, using a sliding hump argument, we may find a subsequence \( \{y'_n\}_{n \in \mathbb{N}} \) and a block \( \{u_n\}_{n \in \mathbb{N}} \) of \( \{x_n\}_{n \in \mathbb{N}} \) which are equivalent. Clearly \( \{u_n\}_{n \in \mathbb{N}} \) is equivalent to the standard basis of \( \ell_1 \). \qed

### 1.5 Shrinking and boundedly complete bases

**Definition 1.5.1.** Let \( X \) be a Banach space with basis \( \{e_n\}_{n \in \mathbb{N}} \). The basis is called shrinking if \( X^* = [e^*_n : n \in \mathbb{N}] \).

Let \( X \) be a Banach space and \( \{x_n\}_{n \in \mathbb{N}} \) be a basic sequence in \( X \). The sequence \( \{x_n\}_{n \in \mathbb{N}} \) is called boundedly complete if for any sequence of reals \( \{\lambda_n\}_{n \in \mathbb{N}} \) such that if

\[
\sup_{n \in \mathbb{N}} || \sum_{i=1}^{n} \lambda_i x_i || < \infty,
\]

then \( \sum_{n=1}^{\infty} \lambda_n x_n \) converges.
Proposition 1.5.1. Let $X$ be a Banach space with basis $\{e_n\}_{n \in \mathbb{N}}$. The following are equivalent:

i) The basis $\{e_n\}_{n \in \mathbb{N}}$ is shrinking

ii) Every unitary block of $\{e_n\}_{n \in \mathbb{N}}$ is weakly null.

Proof. i) $\Rightarrow$ ii) Let $\{u_n\}_{n \in \mathbb{N}}$ be a block with $\|u_n\| = 1 \quad \forall n \in \mathbb{N}$. We will show $u_n \overset{w}{\to} 0$. Let $x^* \in X^*$ with $x^* = \sum_{n=1}^{\infty} \lambda_n e_n^*$. Then for any $\epsilon > 0$, there is $k_0$ such that for any $k > k_0$, we have:

$$\left| \sum_{n=k+1}^{\infty} \lambda_n e_n^* \right| < \epsilon \Rightarrow \left| \sum_{n=k+1}^{\infty} \lambda_n e_n^*(x) \right| < \epsilon \quad \forall \|x\| \leq 1 \quad (1.3)$$

If we consider $u_k = \sum_{i=n_k+1}^{n_{k+1}} \alpha_i e_i$ and $m \in \mathbb{N}$ such that $n_m > k_0$, then for all $k > m$, we have $n_k > k_0$, therefore (1.3) implies that

$$|x^*(u_k)| = \left| \sum_{i=1}^{\infty} \lambda_i e_i^*(u_k) \right| = \left| \sum_{i=n_{k+1}}^{\infty} \lambda_i e_i^*(u_k) \right| < \epsilon \Rightarrow u_k \overset{w}{\to} 0$$

ii) $\Rightarrow$ i) We will argue by contradiction. Assume $\{e_n\}_{n \in \mathbb{N}}$ is not shrinking. Then there exists $x^* \notin [e_n^*: n \in \mathbb{N}]$ with $\|x^*\| = 1$. We observe that $P_n^*(x^*) = \sum_{i=1}^{n} x^*(e_i)e_i^*$ so the sequence $\{P_n^*(x^*)\}_{n \in \mathbb{N}}$ cannot converge in $x^*$. Therefore, there is $\epsilon > 0$ and an increasing sequence of positive integers $\{m_k\}_{k \in \mathbb{N}}$ such that

$$\|x^* - P_{m_k}^*(x^*)\| \geq 2\epsilon \quad \forall k \in \mathbb{N} \quad (1.4)$$

Let $n_1 = m_1$. By (1.4), there is $x_1 = \sum_{n=1}^{\infty} \lambda_n^{(1)} e_n \in X$, with $\|x_1\| = 1$, such that $|x^*(\sum_{i=n_1+1}^{\infty} \lambda_i^{(1)} e_i)| \geq 2\epsilon$. Consider $k_2 \in \mathbb{N}$ such that $m_{k_2} > n_1$ and if we
set \( n_2 = m_{k_2} \), we have

\[
\|x^*\| \cdot \left\| \sum_{i=n_1+1}^{\infty} \lambda_i^{(1)} e_i - \sum_{i=n_1+1}^{n_2} \lambda_i^{(1)} e_i \right\| < \epsilon
\]

\( \Rightarrow |x^*(\sum_{i=n_1+1}^{\infty} \lambda_i^{(1)} e_i)| < |x^*(\sum_{i=n_1+1}^{n_2} \lambda_i^{(1)} e_i)| + \epsilon. \]

We define \( u_1 = \sum_{i=n_1+1}^{n_2} \lambda_i^{(1)} e_i \). Then

\[
|x^*(\sum_{i=n_1+1}^{\infty} \lambda_i^{(1)} e_i)| \geq 2\epsilon \Rightarrow |x^*(u_1)| > \epsilon
\]

By (1.4) there is \( x_2 = \sum_{n=1}^{\infty} \lambda_n^{(2)} e_n \in X \) with \( \|x_2\| = 1 \), such that

\[
|x^*(\sum_{i=n_2+1}^{\infty} \lambda_i^{(1)} e_i)| \geq 2\epsilon.
\]

Consider \( k_3 \in \mathbb{N} \) such that \( m_{k_3} > n_2 \) and if \( n_3 = m_{k_3} \), we have

\[
\|x^*\| \cdot \left\| \sum_{i=n_2+1}^{\infty} \lambda_i^{(2)} e_i - \sum_{i=n_2+1}^{n_3} \lambda_i^{(2)} e_i \right\| < \epsilon,
\]

so

\[
|x^*(\sum_{i=n_1+1}^{\infty} \lambda_i^{(1)} e_i)| < |x^*(\sum_{i=n_2+1}^{n_3} \lambda_i^{(2)} e_i)| + \epsilon
\]

Let us define \( u_2 = \sum_{i=n_2+1}^{n_3} \lambda_i^{(2)} e_i \). Then

\[
|x^*(\sum_{i=n_2+1}^{\infty} \lambda_i^{(2)} e_i)| \geq 2\epsilon \Rightarrow |x^*(u_2)| > \epsilon
\]

Continuing inductively, we construct a block \( \{u_k\}_{k \in \mathbb{N}} \) such that \( |x^*(u_k)| > \epsilon \ \forall k \in \mathbb{N} \). The sequence \( \{u_k\}_{k \in \mathbb{N}} \) clearly is not weakly null. Moreover
\{u_k\}_{k \in \mathbb{N}} is bounded since \|u_k\| = \|P_{n_k+1}(x_k) - P_{n_k}(x_k)\| \leq 2K\|x_k\| = 2K, where \(K\) is the basis constant. Then the sequence \(z_n = \frac{u_n}{\|u_n\|}\) is not null, which is a contradiction. \(\square\)

**Proposition 1.5.2.** Let \(X\) be a Banach space with a shrinking basis \(\{e_n\}_{n \in \mathbb{N}}\) and constant \(K\). Then for any \(x^{**} \in X^{**}\) the following estimate holds:

\[
\frac{1}{K} \sup_{n \in \mathbb{N}} \| \sum_{i=1}^{n} x^{**}(e_i^*)e_i \| \leq \|x^{**}\| \leq \sup_{n \in \mathbb{N}} \| \sum_{i=1}^{n} x^{**}(e_i^*)e_i \|
\]

Clearly, if the basis is monotone, we have \(\|x^{**}\| = \lim_{n \to \infty} \| \sum_{i=1}^{n} x^{**}(e_i^*)e_i \|\).

**Proof.** Let \(n \in \mathbb{N}\). Then

\[
\| \sum_{i=1}^{n} x^{**}(e_i^*)e_i \| = \sup \left\{ \| \sum_{i=1}^{n} x^{**}(e_i^*)y^*(e_i) \| : y^* \in B_{X^*} \right\}
\]

Consider \(y^* \in B_{X^*}\). Since the basis is shrinking we may expand as follows:

\[
y^* = \sum_{n=1}^{\infty} y^*(e_n)e_n^*.
\]

Then we get

\[
\| \sum_{i=1}^{n} x^{**}(e_i^*)y^*(e_i) \| = \| x^{**}(\sum_{i=1}^{n} y^*(e_i)e_i^*) \| \leq \| x^{**} \| \| \sum_{i=1}^{n} y^*(e_i)e_i^* \| \leq K\|x^{**}\|\|y^*\| \leq K\|x^{**}\|,
\]

Hence

\[
\| \sum_{i=1}^{n} x^{**}(e_i^*)e_i \| \leq K\|x^{**}\| \quad \forall n \in \mathbb{N} \Rightarrow \frac{1}{K} \sup_{n \in \mathbb{N}} \| \sum_{i=1}^{n} x^{**}(e_i^*)e_i \| \leq \|x^{**}\|
\]

For the other direction, consider \(x^* \in B_{X^*}\) with \(x^* = \sum_{n=1}^{\infty} x^*(e_n)e_n^*\). Then we
\[ |x^{**}(x^*)| = \left| \sum_{n=1}^{\infty} x^*(e_n)x^{**}(e^*_n) \right| = \lim_{n \to \infty} \left| \sum_{i=1}^{n} x^*(e_i)x^{**}(e^*_i) \right| \leq \sup_{n \in \mathbb{N}} \left| \sum_{i=1}^{n} x^{**}(e^*_i)e_i \right| \leq \sup_{n \in \mathbb{N}} \left| \sum_{i=1}^{n} x^{**}(e^*_i)e_i \right| \]

and the proof is completed. \(\square\)

**Remark 1.5.1.** There is not any subsequence of the standard basis of \(c_0\) which is boundedly complete.

**Proof.** Let \(\{e_{n_k}\}_{k \in \mathbb{N}}\) subsequence of the standard basis of \(c_0\). The sequence \(\{e_{n_k}\}_{k \in \mathbb{N}}\) is clearly basis. We observe that \(\sup_{k \in \mathbb{N}} \|\sum_{i=1}^{k} e_{n_i}\| = 1 < \infty\), while the series \(\sum_{i=1}^{\infty} e_{n_i}\) does not converge. \(\square\)

**Proposition 1.5.3.** Let \(X\) be a Banach space with a boundedly complete bases \(\{x_n\}_{n \in \mathbb{N}}\). Then \(c_0\) does not embed in \(X\).

**Proof.** We will argue by contradiction. Assume there is embedding \(T : c_0 \to X\) and let us denote \(z_n = T(e_n)\), where \(\{e_n\}_{n \in \mathbb{N}}\) is the standard basis of \(c_0\). From embedding, there is \(M > 0\) such that \(\|z_n\| \geq M \ \forall n \in \mathbb{N}\). Moreover we know that \(e_n \xrightarrow{w} 0\), so \(z_n \xrightarrow{w} 0\). By a sliding hump argument, we may find a block \(\{u_n\}_{n \in \mathbb{N}}\) of the basis of \(X\) and a subsequence \(\{z_{n_k}\}_{k \in \mathbb{N}}\) of \(\{z_n\}_{n \in \mathbb{N}}\) which are equivalent. It is easy to see that \(\{u_n\}_{n \in \mathbb{N}}\) is boundedly complete, therefore \(\{x_{n_k}\}_{k \in \mathbb{N}}\) is as well. Since \(T\) is an embedding then \(\{e_{n_k}\}_{k \in \mathbb{N}}\) will be boundedly complete, which is a contradiction. \(\square\)

**Proposition 1.5.4.** Let \(X\) be Banach space with boundedly complete basis \(\{e_n\}_{n \in \mathbb{N}}\) and basis constant \(K\). Then \(X\) is isomorphic to \(Y = [e^*_n, n \in \mathbb{N}]^*\). In particular, if the basis is monotone, we may assume \(T\) is an isometry.

**Proof.** Let us define \(J : X \to Y^*\) by \(J(x)(y) = y(x) \ \forall y \in Y\) It is clear that \(J\) well defined and linear. We also show \(J\) is an embedding. Indeed, let \(x \in X\). Then

\[ |J(x)(y)| = |y(x)| \leq ||x|| \cdot ||y|| \ \forall y \in Y \Rightarrow ||J(x)|| \leq ||x|| \]
In take the other side of the inequality, we make the expansion \( x = \sum_{n=1}^{\infty} \alpha_n e_n \). Denoting \( x_n = \sum_{n=1}^{N} \alpha_n e_n \), we have that \( x = \lim_{n \to \infty} x_n \). We will show \( ||x_n|| \leq K ||J(x_n)|| \) \( \forall n \in \mathbb{N} \) and the claim follows by the continuity of \( J \). Let \( n \in \mathbb{N} \). Then Hahn-Banach Theorem implies that there is \( x^* \in X^* \) with \( ||x^*|| = 1 \) and \( |x^*(x_n)| = ||x_n|| \). We observe that \( x^* \circ P_n \in \{e_1^*, ..., e_n^*\} \subseteq Y \) so \( x_n = P_n(x_n) \). Therefore, we obtain

\[
||x_n|| = |(x^* \circ P_n)(x_n)| = |J(x_n)(x^* \circ P_n)| \leq ||x^*|| \cdot ||P_n|| \cdot ||J(x_n)|| \leq K ||J(x_n)||
\]

It is clear that in the case the basis is monotone we get isometric embedding. It remains to show \( J \) is surjective as well. For this purpose, consider \( y^* \in Y^* \). We define the sequence \( \{\sum_{i=1}^{n} y^*(e_i^*)e_i\}_{n \in \mathbb{N}} \subseteq X \). We show that this sequence is uniformly bounded by \( K^2 ||y^*|| \). Indeed, for any \( n \in \mathbb{N} \) we have

\[
||\sum_{i=1}^{n} y^*(e_i^*)e_i|| \leq K ||J(\sum_{i=1}^{n} y^*(e_i^*)e_i)|| = K ||\sum_{i=1}^{n} y^*(e_i^*)J(e_i)||
\]

It remains to show that \( ||\sum_{i=1}^{n} y^*(e_i^*)J(e_i)|| \leq K ||y^*|| \). Indeed, fixing \( y = \sum_{n=1}^{\infty} y(e_n^*)e_n^* \in Y \) with \( ||y|| \leq 1 \), we obtain

\[
|\sum_{i=1}^{n} y^*(e_i^*)J(e_i)(y)| = |y^*(\sum_{i=1}^{n} y(e_i^*)e_i)| \leq ||y^*|| ||\sum_{i=1}^{n} y(e_i^*)e_i|| \leq ||y^*|| K ||y|| \leq K ||y^*||
\]

Since the basis is boundedly complete the series \( \sum_{i=1}^{\infty} y^*(e_i^*)e_i \) converges. Defining \( x = \sum_{n=1}^{\infty} y^*(e_n^*)e_n \), we have \( y^* = J(x) \) and the proof is complete. \( \square \)

We now give an important characterization of reflexive spaces with Schauder basis.

**Theorem 1.5.1.** (R.C.James) Let \( X \) be a Banach space and \( \{e_n\}_{n \in \mathbb{N}} \) be a basis of \( X \). The following are equivalent

i) The basis \( \{e_n\}_{n \in \mathbb{N}} \) is shrinking and boundedly complete.

ii) \( X \) is reflexive.

**Proof.** i) \( \Rightarrow \) ii) Since the basis is shrinking, we have \( \text{ch} \subseteq [e_n^*, n \in \mathbb{N}]^* \). By Proposition 1.5.4, the canonical embedding, which in this case coincides with
the map $J$ we defined, maps onto $X^{**}$ and so $X$ is reflexive.

(ii) $\Rightarrow$ i) We first show the basis is shrinking. Let $x^* \in X^*$. Then $P_n^*(x^*) = \sum_{i=1}^{n} x^*(e_i)e_i^* \xrightarrow{w^*} x^*$ as $n \to \infty$. Since $X$ is reflexive the $w$-topology coincides with the $w^*$-topology in $X^*$. Therefore

$$P_n^*(x^*) \xrightarrow{w} x^* \Rightarrow X^* = \langle e_{n}, n \in \mathbb{N} \rangle = \langle e_{n}, n \in \mathbb{N} \rangle$$

by Mazur’s Theorem. Therefore, the basis is shrinking.

We also show the basis is boundedly complete. Consider a sequence $\{\alpha_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ such that

$$\sup_{n \in \mathbb{N}} \left\{ || \sum_{i=1}^{n} \alpha_i e_i || \right\} = M.$$  

We will show the series $\sum_{n=1}^{\infty} \alpha_n e_n$ converges. Let us define $x_n = \sum_{i=1}^{n} \alpha_i e_i$ Since $X$ is reflexive the ball $M \cdot B_X$ is $w$-compact. Thus, since $X^*$ is separable, there is $x \in M \Delta B_X$ and subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ such that $x = w - \lim x_{n_k}$. Moreover, for any $i \in \mathbb{N}$ the functional $e_i^*$ is $w$-continuous, so

$$e_i^*(x) = e_i^*(w - \lim x_{n_k}) = \lim_{k \to \infty} e_i^*(x_{n_k}) = \alpha_i \quad \forall i \in \mathbb{N}$$

Therefore, $x = \sum_{n=1}^{\infty} e_n^*(x)e_n = \sum_{n=1}^{\infty} \alpha_n e_n$ and the result follows. 

Before the end of this section, let us give an isometric description of the second conjugate of a Banach space with boundedly complete basis. We first prove two general facts.

**Proposition 1.5.5.** Let $X$ be a normed space and $Y$ be a subspace of $X^{**}$ such that $\hat{X}$ is a subspace of $Y$. Then there is linear isometry $T : X^* \to Y^*$ such that $Y^* = T[X^*] \oplus \hat{X}$. In particular, for $Y = X^{**}$ we get $X^{***} = T[X^*] \oplus \hat{X}$.

**Proof.** Let us define $T : X^* \to Y^*$ by $T(x^*)(y) = y(x^*)$ $\forall y \in Y$. Clearly $T$ is well defined and linear. We also show it is an isometry. Indeed, we have that

$$||T(x^*)|| = \sup \left\{ ||T(x^*)(y)|| : y \in B_Y \right\} = \sup \left\{ |y(x^*)| : y \in B_Y \right\} 
\geq \sup \left\{ |\hat{x}(x^*)| : x \in B_X \right\} = \sup \left\{ |x^*(x)| : x \in B_X \right\} = ||x^*||$$
Moreover
\[ ||T(x^*)|| = \sup \{ |y(x^*)| : y \in B_Y \} \leq \sup \{ |x^{**}(x^*)| : x^{**} \in B_{X^{**}} \} = ||x^*||, \]
hence \( T \) is an isometry. Let us now show that \( T[X^*] \) is complemented subspace of \( Y^* \). Since \( X^* \) is a Banach space, then \( T[X^*] \) is a Banach space and thus closed subspace of \( Y^* \). We define \( Q : Y^* \to X^* \) by \( Q(y^*) = y^* \circ \wedge \), where \( \wedge : X \to X^{**} \) is the canonical embedding of \( X \) in \( X^{**} \). The map \( Q \) is clearly linear and bounded. Let us define \( P : Y^* \to Y^* \) by \( P = T \circ Q \). The map \( P \) is clearly linear and bounded. We also show it is a projection. Indeed, it is straightforward to see \( Q \circ T = I_{Y^*} \), where \( I_{Y^*} \) is the identity map in \( Y^* \). Indeed, \( (Q \circ T)(x^*)(x) = T(x^*)(\hat{x}) = \hat{x}(x^*) = x^*(x) \quad \forall x \in X \). Then \( P^2 = T \circ Q \circ T \circ Q = T \circ Q = P \). It remains to show \( P[Y^*] = T[X^*] \) or in other words \( Q \) is surjective. Consider \( x^* \in X^* \) and define \( y^* = \hat{x}^* | Y \in Y^* \).

Then \( Q(y^*) = x^* \). Therefore \( T[X^*] \) is complemented subspace of \( Y^* \), hence \( X^* = T[X^*] \oplus \ker P \). We finally show \( \ker P = \hat{X}^\perp \). Indeed for any \( y^* \in \hat{X}^\perp \), we get \( P(y^*)(y) = T(Q(y^*)) = T(y^* \circ \wedge)(y) = y(y^* \circ \wedge) = 0 \quad \forall y \in Y \), so \( \hat{X}^\perp \subseteq \ker P \). Conversely, given \( y \in \ker P \) we have that \( T(y^* \circ \wedge) = 0 \) and since \( T \) is injective, we obtain \( y^* \circ \wedge = 0 \), so \( y^* \in \hat{X}^\perp \Rightarrow \ker P \subseteq \hat{X}^\perp \). We finally get \( Y^* = T[X^*] \oplus \hat{X}^\perp \).

**Proposition 1.5.6.** Let \( X \) be a Banach space and \( Y \) a closed subspace. Then \( (X/Y)^* \) isometric to \( Y^\perp \).

**Proof.** We consider the map \( T : (X/Y)^* \to Y^\perp \) given by \( T(\hat{x}^*)(x) = \hat{x}^*(x + Y) \). Then for any \( x \in Y \) we have that \( T(\hat{x}^*)(x) = \hat{x}^*(Y) = 0 \). Moreover, for any \( x \in X \) we have that \( |T(\hat{x}^*)(x)| = |\hat{x}^*(x + Y)| \leq ||\hat{x}^*|| ||y + Y|| \leq ||\hat{x}^*|| ||x|| \) and so \( T \) is well defined. It is also clearly linear as well. We now show \( T \) is surjective isometry. Let \( x^* \in Y^\perp \). We define \( \hat{x}^* : X/Y \to \mathbb{R} \) by \( \hat{x}^*(x + Y) = x^*(x) \). Then for any \( x \in X \), we have that
\[
|\hat{x}^*(x + Y)| = |x^*(x - y)| \quad \forall y \in Y \leq ||x^*|| \cdot ||x - y|| \quad \forall y \in Y \Rightarrow \\
|\hat{x}^*(x + Y)| \leq ||x^*|| \cdot ||x + Y||
\]
so \( \hat{x}^* \in (X/Y)^* \) and clearly \( T(\hat{x}^*) = x^* \) To show it is isometry, consider
\( \hat{x}^* \in (X/Y)^* \). Then

\[
\|T(\hat{x}^*)\| = \sup \{|T(\hat{x}^*)(x)| : x \in B_X\} \leq \sup \{|T(\hat{x}^*)(x)| : x + Y \in B_{X/Y}\} = \sup \{|\hat{x}^*(x + Y)| : x + Y \in B_{X/Y}\} = \|\hat{x}^*\|
\]

For the other inequality, consider \( x \in X \) with \( \|x + Y\| \leq 1 \). Then

\[
|\hat{x}^*(x + Y)| = |T(\hat{x}^*)(x)| = |T(\hat{x}^*)(x - y)|, \quad \forall y \in Y
\]

\[
\leq \|T(\hat{x}^*)\| \cdot \|x - y\|, \quad \forall y \in Y
\]

so

\[
|\hat{x}^*(x + Y)| \leq \|T(\hat{x}^*)\| \cdot \|x + Y\| \leq \|T(\hat{x}^*)\|.
\]

Therefore \( \|T(\hat{x}^*)\| = \|\hat{x}^*\| \) and so \( T \) is surjective isometry.

After those two results, we may give the description we want.

**Proposition 1.5.7.** Let \( X \) be a Banach space with boundedly complete basis \( \{e_n\}_{n \in \mathbb{N}} \) and \( Y = [e_n^*, n \in \mathbb{N}] \). Then we have the decomposition \( X^{**} = \hat{X} \oplus Y^\perp \). Therefore \( \dim(X^{**}/\hat{X}) = \dim Y^\perp = \dim(X^*/Y)^* \).

**Proof.** Using the notation introduced above we get \( Y^{***} = T[Y^*] \oplus \hat{Y}^\perp \). Moreover \( Y^* = J[X] \Rightarrow Y^{***} = J^{**}[X^{**}] \). A simple calculation shows \( J^{**}[\hat{X}] = T[Y^*] \) and \( J^{**}[Y^\perp] = \hat{Y}^\perp \). Then \( X^{**} = \hat{X} \oplus Y^\perp \). Clearly \( (X^{**}/\hat{X}) \cong Y^\perp \equiv (X^*/Y)^* \Rightarrow \dim(X^{**}/\hat{X}) = \dim Y^\perp = \dim(X^*/Y)^* \).
Chapter 2

The James Space

2.1 Definition of $\mathcal{J}$ and elementary properties

For $n, m \in \mathbb{N}$ with $n \leq m$, we will denote $[n, m] = \{k \in \mathbb{N} : n \leq k \leq m\}$. We will call these sets bounded intervals. For $n \in \mathbb{N}$, we will denote $[n, \infty) = \{k \in \mathbb{N} : n \leq k\}$. These sets will be called infinite intervals.

Let us define

$$\mathcal{J} = \left\{ x = \{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R} : \sup \left\{ \sum_{i=1}^{m} \left| \sum_{n \in I_i} x_n \right|^2 \right\} < \infty \right\}$$

where sup is taken over all finite families of disjoint bounded intervals $\{I_i\}_{i=1}^{m}$.

It is straightforward that $\mathcal{J}$ is an infinite dimensional vector space.

For $x \in \mathcal{J}$, let us write

$$\|x\| = \sup \left\{ \sum_{i=1}^{m} \left| \sum_{n \in I_i} x_n \right|^2 \right\}^{1/2},$$

where sup is taken over all finite families of disjoint bounded intervals $\{I_i\}_{i=1}^{m}$.

**Proposition 2.1.1.** $(\mathcal{J}, \| \cdot \|)$ is a Banach space.

**Proof.** We first show that $\| \cdot \|$ is a norm on $\mathcal{J}$. The only non-trivial thing to check is the triangular inequality. Consider $x, y \in \mathcal{J}$ and $\{I_i\}_{i=1}^{m}$ a finite
family of disjoint bounded intervals. Then Minkowski inequality implies

$$
\left( \sum_{i=1}^{m} \left| \sum_{n \in I_i} (x_n + y_n) \right|^2 \right)^{1/2} = \left( \left( \sum_{n \in I_1} x_n + \sum_{n \in I_1} y_n \right)^2 + \cdots + \left( \sum_{n \in I_m} x_n + \sum_{n \in I_m} y_n \right)^2 \right)^{1/2}
\leq \left( \sum_{i=1}^{m} |x_n|^2 \right)^{1/2} + \left( \sum_{i=1}^{m} |y_n|^2 \right)^{1/2} \leq |x| + |y| \Rightarrow |x + y| \leq |x| + |y|.
$$

To show completeness, consider a Cauchy sequence \( \{x^{(n)}\}_{n \in \mathbb{N}} \) on \( J \). Then for any \( \epsilon > 0 \) there is \( N \) such that for all \( m > n > N \), there holds \( |x^{(m)} - x^{(n)}| < \epsilon \). Considering \( i \in \mathbb{N} \) and the trivial interval \( I_i = \{i\} \), definition of the norm implies \( |x^{(m)}_i - x^{(n)}_i| < \epsilon \quad \forall m > n > N \). Therefore the sequence \( \{x^{(n)}_i\}_{n \in \mathbb{N}} \) converges for any \( i \in \mathbb{N} \). We define \( x_i = \lim_{n \to \infty} x^{(n)}_i \) and \( x = \{x_i\}_{i \in \mathbb{N}} \).

We are showing that \( x \in J \) and that \( x = \lim_{n \to \infty} x^{(n)} \). Indeed, consider \( \epsilon > 0 \) and \( M \in \mathbb{N} \), such that for any \( m > n \geq M \), we have \( |x_n - x_m| < \frac{\epsilon}{2} \). Then for any finite family of disjoint, bounded intervals \( \{I_i\}_{i=1}^{k} \), we have

$$
\left( \sum_{i=1}^{k} \left| \sum_{j \in I_i} (x^{(m)}_j - x^{(n)}_j) \right|^2 \right)^{1/2} < \frac{\epsilon}{2} \quad \forall m > n \geq M \quad \Rightarrow \quad (2.1)
$$

$$
\left( \sum_{i=1}^{k} \left| \sum_{j \in I_i} (x^{(n)}_j - x_j) \right|^2 \right)^{1/2} \leq \frac{\epsilon}{2} < \epsilon \quad \forall n \geq M \quad \Rightarrow \quad (2.2)
$$

For \( n = M \) in (2.1), we get \( x^{(M)} - x \in J \Rightarrow x \in J \) and \( x = \lim_{n \to \infty} x^{(n)} \) \( \square \)

From now on, for any \( n \in \mathbb{N} \), we will denote \( e_n = X_{\{n\}} \).

**Remark 2.1.1.** Let \( x \in J \). Consider \( k \in \mathbb{N} \) and define \( s_k = \sum_{i=1}^{k} x_i e_i \). Then there holds the estimate

$$
||s_k||^2 + ||x - s_k||^2 \leq ||x||^2
$$

**Proof.** Let \( \{I_i\}_{i=1}^{m} \) bounded intervals and \( l \in \{1, \ldots, m - 1\} \) such that \( I_i \subseteq
∀i = 1, ..., l and \( I_i \subseteq [n+1, +\infty) \) ∀i = l + 1, ..., m. Then

\[
\sum_{i=1}^{l} |\sum_{n \in I_i} s_n|^2 + \sum_{i=l+1}^{m} |\sum_{n \in I_i} (x-s_n)|^2 = \sum_{i=1}^{m} |\sum_{n \in I_i} x_n|^2 \leq ||x||^2 \Rightarrow \\
||s_k||^2 + ||x-s_k||^2 \leq ||x||^2
\]

Proposition 2.1.2. The sequence \( \{e_n\}_{n \in \mathbb{N}} = (X_{\{n\}})_{n \in \mathbb{N}} \) is monotone and unitary Schauder basis of \( J \).

Proof. Clearly, \( e_n \neq 0 \) ∀n ∈ \( \mathbb{N} \) and \( ||e_n|| = 1 \) ∀n ∈ \( \mathbb{N} \). We first show that \( J = [e_n : n \in \mathbb{N}] \). Consider \( x = \{x_n\}_{n \in \mathbb{N}} \in J \). Let us define \( s_n = \sum_{i=1}^{n} x_i e_i \).

We will show \( x = \lim_{n \to \infty} s_n \). Indeed, consider \( \epsilon > 0 \). Then the norm definition implies that there is a finite family of disjoint bounded intervals \( \{I_i\}_{i=1}^{m} \) such that

\[
\sum_{i=1}^{m} |\sum_{n \in I_i} x_n|^2 > ||x||^2 - \epsilon^2 \tag{2.3}
\]

Setting \( n_0 = \max \left\{ n : n \in \bigcup_{i=1}^{m} I_i \right\} \) in (2.3) we get

\[
||x||^2 - ||s_n||^2 < \epsilon^2 \quad \forall n \geq n_0
\]

Thus for \( n \geq n_0 \) we get \( ||x-s_n||^2 = ||x-s_n||^2 + ||s_n||^2 - ||s_n||^2 \leq ||x||^2 - ||s_n||^2 < \epsilon^2 \), by Remark 2.1.1. Hence, \( x = \lim_{n \to \infty} s_n \). Finally, for \( k, n \in \mathbb{N} \) with \( k > n \) and \( \alpha_1, ..., \alpha_k \in \mathbb{R} \), we consider disjoint and bounded intervals \( \{I_i\}_{i=1}^{m} \) such that \( I_i \subseteq [1,n] \) ∀i = 1, ..., m. Then

\[
\left( \sum_{i=1}^{m} |\sum_{j \in I_i} \alpha_j e_j|^2 \right)^{1/2} \leq \left| \sum_{i=1}^{k} \alpha_i e_i \right| \Rightarrow \left| \sum_{i=1}^{n} \alpha_i e_i \right| \leq \left| \sum_{i=1}^{k} \alpha_i e_i \right|
\]

and the result follows.

Proposition 2.1.3. The basis \( \{e_n\}_{n \in \mathbb{N}} \) of \( J \) is boundedly complete. Therefore \( c_0 \) does not embedded in \( J \).
Proof. Consider a sequence \( \{\alpha_n\}_{n \in \mathbb{N}} \) of real numbers such that \( \sup_{n \in \mathbb{N}} \left\| \sum_{i=1}^{n} \alpha_i e_i \right\| < \infty \). Since \( \{e_n\}_{n \in \mathbb{N}} \) is monotone, we get

\[
\lim_{n \to \infty} \left\| \sum_{i=1}^{n} \alpha_i e_i \right\| = \sup_{n \in \mathbb{N}} \left\| \sum_{i=1}^{n} \alpha_i e_i \right\|
\]

so the sequence \( \left\{ \left\| \sum_{i=1}^{n} \alpha_i e_i \right\|^2 \right\}_{n \in \mathbb{N}} \) is Cauchy. Hence, for any \( \epsilon > 0 \), there is \( n_0 \in \mathbb{N} \) such that for any \( m > n > n_0 \) we have \( \left\| \sum_{i=1}^{m} \alpha_i e_i \right\|^2 - \left\| \sum_{i=1}^{n} \alpha_i e_i \right\|^2 < \epsilon^2 \). Therefore by Remark 2.1.1 we get

\[
\left\| \sum_{i=n+1}^{m} \alpha_i e_i \right\| \leq \sqrt{\left\| \sum_{i=1}^{m} \alpha_i e_i \right\|^2 - \left\| \sum_{i=1}^{n} \alpha_i e_i \right\|^2} < \epsilon.
\]

Since \( \mathcal{J} \) is a Banach space, the series \( \sum_{n=1}^{\infty} \alpha_n e_n \) converges. \( \square \)

Remark 2.1.2. If we restrict the norm of \( \mathcal{J} \) in \( < c_{00}(\mathbb{N}) > \) then \( c_{00}(\mathbb{N}) \) is not complete since it has countable algebraic basis. It is clear that \( \mathcal{J} \) is the completion \( < c_{00}(\mathbb{N}) > \) under this norm. This is an alternative definition of \( \mathcal{J} \).

Remark 2.1.3. Let \( I \) be a bounded interval. Let us define \( I^* = \sum_{n \in I} e^*_n \). It is clear that \( I^* \in \mathcal{J}^* \). Consider now an infinite interval \( I \). The definition of the norm implies that for any \( x \in \mathcal{J} \) and \( \epsilon > 0 \), there is \( n_0 \in \mathbb{N} \) such that for any \( m > n > n_0 \), we get \( \left| \sum_{i=n+1}^{m} e^*_i(x) \right| < \epsilon \). Therefore, the sequence \( \sum_{n \in I} e^*_n(x) \) converges for any \( x \in \mathcal{J} \). Defining \( I^* : \mathcal{J} \to \mathbb{R} \) by \( I^*(x) = \sum_{n \in I} e^*_n(x) \) \( \forall x \in \mathcal{J} \), we obtain that \( I^* \) is linear and by Banach-Steinhauss Theorem, we get \( I^* \in \mathcal{J}^* \). It is clear that \( I^* \) w* \( \sum_{n \in I} e^*_n \) and \( \left\| I^* \right\| = 1 \) for any interval \( I \).
Let us denote
\[ s^* \equiv \sum_{n=1}^{\infty} e_n^*. \]
It is easy to see that \( s^* \notin [e_n^* : n \in \mathbb{N}] \). Indeed, if this was not the case we would have \( s^* \| \| \sum_{n=1}^{\infty} e_n^* \), so for any \( 0 < \epsilon < 1 \) there would be \( n_0 \in \mathbb{N} \) such that
\[ |s^*(x) - \sum_{i=1}^{n_0} e_n^*(x)| < \epsilon \quad \forall x \in B, \quad x = e_{n_0+1} \Rightarrow 1 < \epsilon, \]
which is a contradiction. Clearly \( I^* \notin [e_n^* : n \in \mathbb{N}] \) for any infinite interval \( I \). Therefore the basis defined is not and so Theorem 1.5.1 implies the space \( J \) is not reflexive.

If we consider \( x = \sum_{n=1}^{\infty} \alpha_n e_n \in J \) and \( x_n = \sum_{i=1}^{n} \alpha_i e_i \), we get
\[ ||x_n|| = \sup \left\{ \sum_{i=1}^{m} |I_i^*(x_n)|^2 \right\}^{1/2} \quad \forall n \in \mathbb{N} \quad (2.4) \]
where \( \{I_i\}_{i=1}^{m} \) are disjoint intervals which we can assume they are infinite as well since \( \text{supp} x_n \) is finite for any \( n \in \mathbb{N} \). Letting \( n \to \infty \) in (2.4) we get the following description for the norm
\[ ||x|| = \sup \left\{ \sum_{i=1}^{m} |I_i^*(x)|^2 \right\}^{1/2} \]
where \( \{I_i\}_{i=1}^{m} \) are disjoint, which can be infinite as well.

### 2.2 Description of \( J^* \)

The goal of this section is to give description of \( J^* \). We will show that \( J^* \) is separable and in particular \( J^* = [e_n^*, n \in \mathbb{N}] \oplus [s^*] \). To see all these properties we will use a different basis for \( J \).

**Proposition 2.2.1.** Consider the sequence of increments \( \{d_n\}_{n \in \mathbb{N}} \) defined by:
\[ d_1 = e_1 \text{ and } d_n = e_n - e_{n-1}, n > 1. \] Then \( \{d_n\}_{n \in \mathbb{N}} \) is a monotone Schauder basis for \( J \).
Proof. It is obvious that $<d_n, n \in \mathbb{N}> = <e_n, n \in \mathbb{N}> \Rightarrow [d_n, n \in \mathbb{N}] = \mathcal{J}$.
Consider $n \in \mathbb{N}$ and $\alpha_1, ..., \alpha_n, \alpha_{n+1} \in \mathbb{R}$ and a family of disjoint intervals
$
\{I_i\}_{i=1}^m$ of $[1, n]$. If $n \notin \bigcup_{i=1}^m I_i$, then

$$
\left( \sum_{i=1}^m |I_i^* \left( \sum_{i=1}^n \alpha_i d_i \right) |^2 \right)^{1/2} = \left( \sum_{i=1}^m |I_i^* \left( \sum_{i=1}^{n+1} \alpha_i d_i \right) |^2 \right)^{1/2} \leq \left\| \sum_{i=1}^{n+1} \alpha_i d_i \right\| \Rightarrow \left\| \sum_{i=1}^{n+1} \alpha_i d_i \right\| \leq \left\| \sum_{i=1}^m \alpha_i d_i \right\|
$$

Otherwise, without loss of generality, we may assume that $I_m = [i, n]$, for some $i \leq n$. Then

$$
\left( \sum_{i=1}^m |I_i^* \left( \sum_{i=1}^n \alpha_i d_i \right) |^2 \right)^{1/2} = \left( \sum_{i=1}^{m-1} |I_i^* \left( \sum_{i=1}^n \alpha_i d_i \right) |^2 + |I_m^* \left( \sum_{i=1}^n \alpha_i d_i \right) |^2 \right)^{1/2}
= \left( \sum_{i=1}^{m-1} |I_i^* \left( \sum_{i=1}^n \alpha_i d_i \right) |^2 + \alpha_{n+1}^2 \right)^{1/2} \leq \left\| \sum_{i=1}^{n+1} \alpha_i d_i \right\|
$$

so

$$
\left\| \sum_{i=1}^n \alpha_i d_i \right\| \leq \left\| \sum_{i=1}^{n+1} \alpha_i d_i \right\|
$$

Therefore, in any case we have that $\left\| \sum_{i=1}^n \alpha_i d_i \right\| \leq \left\| \sum_{i=1}^{n+1} \alpha_i d_i \right\|$ and so
$
\{d_n\}_{n \in \mathbb{N}}$ is a monotone Schauder basis.

\[ \square \]

**Proposition 2.2.2.** The basis $\{d_n\}_{n \in \mathbb{N}}$ is shrinking.

We first prove the claim: If $\{u_n\}_{n \in \mathbb{N}}$ is block of $\{d_n\}_{n \in \mathbb{N}}$ with $M = \sup_{n \in \mathbb{N}} \|u_n\|$, then the series $\sum_{n=1}^\infty \frac{u_n}{n}$ converges.

Proof of the claim:

We will show that for any $m > k \in \mathbb{N}$ we have that $\left\| \sum_{i=k}^m \frac{u_i}{i} \right\|^2 \leq 5M^2 \sum_{i=k}^m \frac{1}{i^2}$
and the series will clearly converge since $\mathcal{J}$ is a Banach space. Denoting
Consider a finite family of disjoint intervals \( \{I_j\}_{j=1}^\ell \). The \( \{1, ..., \ell\} \) is partitioned in the following subsets:

\[
F_k = \{ j \in \{1, ..., \ell\} : I_j \subseteq [n_k, n_k+1-1]\}
\]

\[
F_s = \{ j \in \{1, ..., \ell\} : I_j \subseteq [n_s+1, n_{s+1}-1]\}, s = k+1, ..., m-1
\]

\[
F_m = \{ j \in \{1, ..., \ell\} : I_j \subseteq [n_m+1, n_m+1]\}
\]

\[
F = \bigcup_{i=k}^m F_i
\]

\[
U = \{ j \in \{1, ..., \ell\} : \exists s_1 < s_2 \in \{k+1, ..., m\} : I_j \cap \text{supp}(u_{s_1}) \neq \emptyset, I_j \cap \text{supp}(u_{s_2}) \neq \emptyset \}
\]

Without loss of generality, we may assume \( F, U \) partition the set \( \{1, ..., \ell\} \). Then for any \( s = k, k+1, ..., m \) such that \( F_s \neq \emptyset \), we get

\[
\sum_{j \in F_s} |I_j^*(\sum_{i=k}^m u_i)|^2 \leq \frac{1}{s^2} \sum_{j \in F_s} |I_j^*(u_s)|^2 \leq \frac{M^2}{s^2} \Rightarrow \sum_{j \in F} |I_j^*(\sum_{i=k}^m u_i)|^2 \leq M^2 \sum_{i=k}^m \frac{1}{i^2}
\]

For \( j \in U \), we define

\[
s_{j,1} = \min \{ i \in \{1, ..., m\} : I_j \cap \text{supp}(u_i) \neq \emptyset \}
\]

\[
s_{j,2} = \max \{ i \in \{1, ..., m\} : I_j \cap \text{supp}(u_i) \neq \emptyset \}
\]

Then

\[
|I_j^*(\sum_{i=k}^m u_i)|^2 = |I_j^*(u_{s_{j,1}} + u_{s_{j,2}})|^2 \leq \frac{2M^2}{s_{j,1}^2} + \frac{2M^2}{s_{j,2}^2} \leq \frac{4M^2}{s_{j,1}^2}
\]
Therefore, since $|U| \leq m - k + 1$, we have

$$
\sum_{j \in U} |I_j^* (\sum_{i=k}^{m} \frac{u_i}{i})|^2 \leq 4M^2 \sum_{i=k}^{m} \frac{1}{i^2}
$$

Putting all pieces together we have $\left| \sum_{i=k}^{m} \frac{u_i}{i} \right|^2 \leq 5 M^2 \sum_{i=k}^{m} \frac{1}{i^2}$, hence the series $\sum_{n=1}^{\infty} \frac{u_n}{n}$ converges. Let $x = \sum_{n=1}^{\infty} \frac{u_n}{n}$.

Proof. Assuming $(d_n)_{n \in \mathbb{N}}$ is not shrinking, the proof of Proposition 1.5.1 implies there are $x^* \in X^*$, $\epsilon > 0$ and $(u_n)_{n \in \mathbb{N}}$ block of $(d_n)_{n \in \mathbb{N}}$ such that $x^*(u_n) > \epsilon$, for any $n \in \mathbb{N}$. Then $x^*(x) = \sum_{n=1}^{\infty} \frac{x^*(u_n)}{n} > \epsilon \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$, which is a contradiction. Hence the basis $(d_n)_{n \in \mathbb{N}}$ is shrinking.

Theorem 2.2.1. The conjugate space decomposes as follows:

$$
J^* = [e_n^*, n \in \mathbb{N}] \oplus < s^* >.
$$

Clearly $J^*$ is separable, hence $\ell_1$ does not embed in $J$.

Proof. It is clear that $< d_n^*, n \in \mathbb{N} > = [e_n^*, n \in \mathbb{N}] \oplus < s^* >$. Therefore

$$
J^* = [d_n^*, n \in \mathbb{N}] = [e_n^*, n \in \mathbb{N}] \oplus < s^* > 
\subseteq [e_n^*, n \in \mathbb{N}] \oplus < s^* > = [e_n^*, n \in \mathbb{N}] \oplus < s^* >
$$

since $\dim < s^* > = 1 < +\infty$. Hence, $J^* = [e_n^*, n \in \mathbb{N}] \oplus < s^* >$. \qed

Remark 2.2.1. Both $\ell_1$ and $c_0$ do not embed in $J$. Therefore, since $J$ is not reflexive, no basis of $J$ can be unconditional (see [7] for unconditional bases).

2.3 Description of $J^{**}$

We have seen that $J$ is not reflexive. In this section we will find a non-trivial element of $J^{**}$ and we will give an isometric description for the second conjugate. We will then show that $J$ is isomorphic with the classical space $J'$ which was originally created by R.C.James and isometric with its second conjugate, under an appropriate norm. Therefore $J$ will be isomorphic with $J^{**}$. 41
Proposition 2.3.1. There exists $e^{**} \in J^{**} \setminus \hat{J}$ with $e_n \overset{w^*}{\to} e^{**}$ and $[e_n^*, n \in \mathbb{N}]^\perp = [e^{**}]$

Proof. We first show that $\{e_n\}_{n \in \mathbb{N}}$ is $w$-Cauchy. Indeed, consider $x^* \in J^*$. By Theorem 2.2.1, we may assume $x^* = \sum_{n=1}^{\infty} \lambda_n e_n^* + \lambda s^*$. Then

$$x^*(e_n) = \lambda_n + \lambda \overset{n \to \infty}{\to} \lambda$$

since $\lambda_n \overset{n \to \infty}{\to} 0$. Banach-Steinhaus Theorem implies that there is $e^{**} \in J^{**}$ with $e_n \overset{w^*}{\to} e^{**}$. It is clear that $e^{**} \notin \hat{J}$. Indeed, otherwise $e^{**}$ would be $w^*$-continuous and $e^{**}(s^*) = \sum_{n=1}^{\infty} e^{**}(e_n^*) = 0$. But $e^{**}(s^*) = \lim_{n \to \infty} s^*(e_n) = 1$.

We finally show that $[e_n^*, n \in \mathbb{N}]^\perp = [e^{**}]$. Indeed, let $x^* = \sum_{n=1}^{\infty} \lambda_n e_n^* \in [e_n^*, n \in \mathbb{N}]$. Then

$$e^{**}(\sum_{n=1}^{\infty} \lambda_n e_n^*) = \sum_{n=1}^{\infty} \lambda_n e^{**}(e_n^*) = 0 \Rightarrow <e^{**}> \subseteq [e_n^*, n \in \mathbb{N}]^\perp$$

Conversely, let $x^{**} \in [e_n^*, n \in \mathbb{N}]^\perp$ and $x^* \in J^*$. Then there are unique $y^* \in [e_n^*, n \in \mathbb{N}]$ and $\lambda \in \mathbb{R} : x^* = y^* + \lambda s^* \Rightarrow x^{**}(x^*)$

$$= \lambda x^{**}(s^*) = x^{**}(s^*)e^{**}(x^*)$$

$$\Rightarrow x^{**} = x^{**}(s^*)e^{**}$$

Thus, $[e_n^*, n \in \mathbb{N}]^\perp \subseteq <e^{**}>$. Combining all the above, we obtain $[e_n^*, n \in \mathbb{N}]^\perp = <e^{**}>$. □

Theorem 2.3.1. We have the following description:

$$J^{**} = \hat{J} \oplus [e^{**}]$$

By this description it is clear that $J^{**}$ is separable and 1-codimension quasi-reflexive, meaning $\dim(J^{**}/\hat{J}) = 1$.

Proof. Since $\{e_n\}_{n \in \mathbb{N}}$ is boundedly complete basis, Proposition 1.5.7 and Proposition 2.3.1 imply the result. □
We now define the classical James space, which is denoted by \( \mathcal{J}' \) and we equip it with two norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) respectively. We will show these two norms are equivalent, that \( \mathcal{J} \) is isometric with \( (\mathcal{J}', \| \cdot \|_1) \) and that \( (\mathcal{J}', \| \cdot \|_2) \) is isometric with its second conjugate. Therefore \( \mathcal{J} \) will be isomorphic with its second conjugate.

Let us define
\[
\mathcal{J}' = \left\{ \{x_n\} \in c_0 : \sup \left\{ (x_{p_1} - x_{p_2})^2 + (x_{p_2} - x_{p_3})^2 + \ldots + (x_{p_{m-1}} - x_{p_m})^2 \right\} < \infty \right\}
\]
where sup is taken for all \( m \in \mathbb{N} \) and positive integers \( p_1 < \ldots < p_m \).

For \( x \in \mathcal{J}' \), we define
\[
\|x\|_1 = \sup \left\{ (x_{p_1} - x_{p_2})^2 + \ldots + (x_{p_m - 1} - x_{p_m})^2 : m \in \mathbb{N}, p_1 < \ldots < p_m \in \mathbb{N} \right\}^{1/2}
\]
With similar arguments to the case of \( \mathcal{J} \) we may show that \( (\mathcal{J}', \| \cdot \|_1) \) is a Banach Banach and that the sequence \( \{e_n \}_{n \in \mathbb{N}} \) given by \( e_n = X_{\{n\}} \) is a monotone Schauder basis of \( (\mathcal{J}', \| \cdot \|_1) \).

If
\[
\|x\|_2 = \sup \left\{ (x_{p_1} - x_{p_2})^2 + \ldots + (x_{p_{m-1}} - x_{p_m})^2 + (x_{p_m} - x_{p_1})^2 \right\}^{1/2}
\]
where sup is taken for all \( m \in \mathbb{N} \) and positive integers \( p_1 < \ldots < p_m \), we can easily see that \( \frac{1}{\sqrt{2}} \|x\|_2 \leq \|x\|_1 \leq \|x\|_2 \). Therefore these two norms are equivalent and \( (\mathcal{J}', \| \cdot \|_2) \) is a Banach space with the same Schauder basis. With similar arguments, one can see that the basis is monotone for \( \| \cdot \|_2 \) as well.

**Proposition 2.3.2.** The Banach spaces \( (\mathcal{J}, \| \cdot \|) \) and \( (\mathcal{J}', \| \cdot \|_1) \) are isometric.

**Proof.** By Proposition 1.3.1, it suffices to show that for any \( n \in \mathbb{N} \) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \), we have that \( \| \sum_{i=1}^{n} \alpha_i d_i \| = \| \sum_{i=1}^{n} \alpha_i e_i \|_1 \). Indeed, consider \( m \in \mathbb{N} \) and the positive integers \( 1 \leq p_1 \leq p'_1 < p_2 \leq p'_2 < \ldots < p_m \leq p'_m \leq n \). Considering the intervals \( I_j = [p_j, p'_j] \) \( \forall j = 1, \ldots, m \) and setting \( \alpha_{n+1} = 0 \), we obtain that
\[
\sum_{j=1}^{m} |I_j| \left( \sum_{i=1}^{n} \alpha_i d_i \right)^2 = (\alpha_{p_1} - \alpha_{p'_1})^2 + (\alpha_{p_2} - \alpha_{p'_2})^2 + \ldots + (\alpha_{p_m} - \alpha_{p'_m})^2
\]
\[
\leq \| \sum_{i=1}^{n+1} \alpha_i e_i \|_1^2 = \| \sum_{i=1}^{n} \alpha_i e_i \|_1^2 \Rightarrow \| \sum_{i=1}^{n} \alpha_i d_i \| \leq \| \sum_{i=1}^{n} \alpha_i e_i \|_1
\]

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Without loss of generality, let us consider \( m \in \mathbb{N} \) and \( p_1 < \ldots < p_m \leq n + 1 \). Then for \( I_j = [p_j, p_{j+1} - 1], j = 1, \ldots, m \) we get

\[(\alpha_{p_1} - \alpha_{p_2})^2 + \ldots + (\alpha_{p_{m-1}} - \alpha_{p_m})^2 = \sum_{j=1}^{m} |I_j^*| \leq \left\| \sum_{i=1}^{n} \alpha_i d_i \right\|^2 \Rightarrow \]

\[\Rightarrow \left\| \sum_{i=1}^{n} \alpha_i e_i \right\|_1 \leq \left\| \sum_{i=1}^{n} \alpha_i d_i \right\| \]

and the result is proved.

\[\square\]

**Proposition 2.3.3.** The basis \( \{e_n\}_{n \in \mathbb{N}} \) is shrinking both for \((\mathcal{J}', \|\cdot\|_1)\) and \((\mathcal{J}', \|\cdot\|_2)\).

**Proof.** Since 

\[\| \sum_{i=1}^{n} \alpha_i d_i \| = \left\| \sum_{i=1}^{n} \alpha_i e_i \right\|_1 \quad \forall \alpha_1, \ldots, \alpha_n \in \mathbb{R}, n \in \mathbb{N}\]

and the norms \( \| \cdot \|_1, \| \cdot \|_2 \), the claim in Proposition 2.2.2 implies that for any block \( \{u_n\}_{n \in \mathbb{N}} \) of \( \{e_n\}_{n \in \mathbb{N}} \) the series \( \sum_{n=1}^{\infty} \frac{u_n}{n} \) converges. The claim comes as in Proposition 2.2.2.

\[\square\]

**Proposition 2.3.4.** For any \( x^{**} \in \mathcal{J}^{**} \) the sequence \( \{x^{**(e_n^*)}\}_{n \in \mathbb{N}} \) converges.

**Proof.** Let \( x^{**} \in \mathcal{J}^{**} \). We will use \( \|\cdot\|_1 \). Since the basis \( \{e_n\}_{n \in \mathbb{N}} \) is monotone and shrinking, Proposition 1.5.2 yields that 

\[\|x^{**}\|_1 = \lim_{n \to \infty} \left\| \sum_{i=1}^{n} x^{**(e_i^*)} e_i \right\|_1\]

For \( n \in \mathbb{N} \), let us define \( x_n = \sum_{i=1}^{n} x^{**(e_i^*)} e_i \). Then there is \( N \in \mathbb{N} \) such that 

\[\left\| x_n \right\|^2 - \left\| x_N \right\|^2 < \epsilon^2 \quad (2.5)\]

Moreover, there is \( k \in \mathbb{N} \) and positive integers \( p_1 < \ldots < p_k \leq N + 1 \) such that 

\[\left\| x_N \right\|_1 = \left[ x^{**(e_{p_1}^*)} - x^{**(e_{p_2}^*)} \right]^2 + \ldots + \left[ x^{**(e_{p_{k-1}}^*)} - x^{**(e_{p_k}^*)} \right]^2.\]
Then for any \( m > n \geq N + 2 \), we obtain \( |x^{**}(e_m) - x^{**}(e_n)| < \epsilon \), hence the sequence \((x^{**}(e_n))_{n \in \mathbb{N}}\) is Cauchy so it converges. Indeed, if not, there would be \( m \in \mathbb{N} \) with \( ||x_m||_1 > ||x^{**}||_1 \) which leads to contradiction, since
\[
||x^{**}||_1 = \sup_{n \in \mathbb{N}} ||\sum_{i=1}^{n} x^{**}(e_i^*)e_i||_1
\]
\[\square\]

**Theorem 2.3.2.** \((\mathcal{J}', || \cdot ||_2)\) is isometric with its second conjugate.

**Proof.** Let us define \( \pi : \mathcal{J}'' \rightarrow \mathcal{J}' \), by \( \pi(x^{**}) = (-\lambda, x^{**}(e_1^*) - \lambda, x^{**}(e_2^*) - \lambda, ...) \) where \( \lambda = \lim_{n \to \infty} x^{**}(e_n^*) \), whose existence is guaranteed by Proposition 2.3.4. It is straightforward to see that \( \pi \) is well defined and linear. Defining \( \alpha_1 = -\lambda, ..., \alpha_n = x^{**}(e_{n-1}^*) - \lambda \), we get \( \pi(x^{**}) = \sum_{n=1}^{\infty} \alpha_n e_n \), so
\[
||\pi(x^{**})||_2 = ||\sum_{n=1}^{\infty} \alpha_n e_n||_2 = \sup_{n \in \mathbb{N}} ||\sum_{i=1}^{n} \alpha_i e_i||_2 = \sup_{n \in \mathbb{N}} ||\sum_{i=1}^{n} x^{**}(e_i^*)e_i||_2 = ||x^{**}||_2
\]
since \( \{e_n\}_{n \in \mathbb{N}} \) is monotone and shrinking basis of \((\mathcal{J}', || \cdot ||_2)\). We also show it is surjective. Let \( x = \{x_n\}_{n \in \mathbb{N}} \in \mathcal{J}' \). Then one can see that \( \sup_{n \in \mathbb{N}} ||\sum_{i=1}^{n} (x_{i+1} - x_1)e_i||_2 < \infty \). Let us define \( y_n = \sum_{i=1}^{n} (x_{i+1} - x_1)e_i \). Since \( \{y_n\}_{n \in \mathbb{N}} \) is bounded and \( \mathcal{J}^* \) is separable, Alaoglu’s Theorem implies there is \( x^{**} \in \mathcal{J}'' \) and a subsequence \( \{y'_n\}_{n \in \mathbb{N}} \) of \( \{y_n\}_{n \in \mathbb{N}} \) with \( \hat{y}'_n \xrightarrow{\text{weak}} x^{**} \Rightarrow x^{**}(e_n^*) = x_{n+1} - x_1 \quad \forall n \in \mathbb{N} \). One can easily verify that \( \pi(x^{**}) = x \). \[\square\]

**Remark 2.3.1.** Since \( \mathcal{J} \) is isomorphic with \((\mathcal{J}', || \cdot ||_2)\), \( \mathcal{J} \) is isomorphic with \( \mathcal{J}^{**} \), while \((\mathcal{J}', || \cdot ||_2)\) is not reflexive.

### 2.4 \( \mathcal{J} \) is \( \ell_2 \)-saturated

In this section, we will see that \( \mathcal{J} \) is \( \ell_2 \)-saturated i.e. each of its infinite dimensional subspaces isomorphically contains \( \ell_2 \). Due to isomorphism \( \mathcal{J}' \) shares the same property under both norms we have defined.
Remark 2.4.1. Let \( \{u_n\}_{n \in \mathbb{N}} \) be a unitary block of \( \{e_n\}_{n \in \mathbb{N}} \). then for any \( m \in \mathbb{N} \) and \( \alpha_1, ..., \alpha_m \in \mathbb{R} \), we have that
\[
\left( \sum_{i=1}^{m} \alpha_i^2 \right)^{1/2} \leq \left\| \sum_{i=1}^{m} \alpha_i u_i \right\|
\]

Proof. Without loss of generality, we consider \( m \in \mathbb{N} \) and \( \alpha_1, ..., \alpha_m \in \mathbb{R} \) such that
\[
\sum_{i=1}^{m} \alpha_i^2 = 1.
\]
We will show
\[
1 \leq \left\| \sum_{i=1}^{m} \alpha_i u_i \right\|.
\]
Indeed, the fact that \( \|u_i\| = 1 \), for any \( i = 1, ..., m \) and the fact that \( \{u_n\}_{n \in \mathbb{N}} \) is block imply there are intervals \( \{I_j\}_{j=1}^{\ell} \) disjoint and bounded and \( 1 < \ell_1 < ... < \ell_{m-1} < \ell \) such that, if we denote \( \ell_0 = 1 \) and \( \ell_m = \ell \), we get \( \{I_j\}_{j=\ell_{i-1}}^{\ell_i} \subseteq \text{supp}(u_i) \) \( \forall i = 1, ..., m \) and
\[
\sum_{j=\ell_{i-1}}^{\ell_i} |I_j^*(u_i)|^2 = 1 \quad \forall i = 1, ..., m,
\]
so
\[
\sum_{j=1}^{\ell} |I_j^*(\sum_{i=1}^{m} \alpha_i u_i)|^2 = \sum_{i=1}^{m} \alpha_i^2 = 1 \Rightarrow \left\| \sum_{i=1}^{m} \alpha_i u_i \right\| \leq 1
\]

Lemma 2.4.1. Let \( \{u_n\}_{n \in \mathbb{N}} \) unitary block of \( \{e_n\}_{n \in \mathbb{N}} \) with \( \lim_{n \to \infty} s^*(u_n) = 0 \). Then for any \( \epsilon > 0 \), there is subsequence \( \{u'_n\}_{n \in \mathbb{N}} \) such that for any \( m \in \mathbb{N} \) and \( \alpha_1, ... \alpha_m \in \mathbb{R} \), there holds
\[
(1 - \epsilon/2)(\sum_{i=1}^{m} \alpha_i^2)^{1/2} \leq \left\| \sum_{i=1}^{m} \alpha_i u_i' \right\| \leq (\sqrt{5} + \epsilon/2)(\sum_{i=1}^{m} \alpha_i^2)^{1/2}
\]

Proof. Assuming \( u_k = \sum_{i=n_k+1}^{n_{k+1}} \lambda_i e_i \), let us define \( y_k = u_k - s^*(u_k)e_{n_k+1} \). So
\[
\|y_k\| \leq 1 + |s^*(u_k)| \quad \forall k \in \mathbb{N} \text{ and } \lim_{k \to \infty} \|u_k - y_k\| = 0,
\]
so we may find a
subsequence \( \{ y_n' \} \) of \( (y_n) \) such that, if \( \{ u_n' \} \) is the corresponding subsequence of \( \{ u_n \} \), we have

\[
\sum_{k=1}^{\infty} \| u_k' - y_k' \|^2 < \frac{\epsilon^2}{16} \quad \text{and} \quad \| y_k' \| < 1 + \frac{\epsilon^2}{128} \quad \forall k \in \mathbb{N}
\]

Consider \( m \in \mathbb{N} \) and \( \alpha_1, \ldots, \alpha_m \in \mathbb{R} \) with \( \sum_{i=1}^{m} \alpha_i^2 = 1 \). Then last Remark implies

\[
1 - \frac{\epsilon}{2} < 1 \leq \left\| \sum_{i=1}^{m} \alpha_i u_i' \right\|
\]

To obtain the other direction of the inequality, we show \( \left\| \sum_{i=1}^{m} \alpha_i y_i' \right\| \leq \sqrt{5 + \frac{\epsilon}{4}} \)

Indeed, consider disjoint bounded intervals \( \{ I_j \} \). For any \( i = 1, \ldots, m \), we denote \( F_i = \{ j \in \{1, \ldots, \ell \} : I_j \subseteq \text{supp}(y_i') \} \) and \( F = \bigcup_{i=1}^{m} F_i \). Then

\[
\sum_{j \in F} |I_j^* (\sum_{i=1}^{m} \alpha_i y_i')|^2 = \sum_{i: F_i \not= \emptyset} \sum_{j \in F_i} |I_j^* (\alpha_i y_i')|^2 \leq \sum_{i: F_i \not= \emptyset} \alpha_i^2 (1 + \frac{\epsilon^2}{128}) < 1 + \frac{\epsilon^2}{32}
\]

We also consider

\[
U = \{ j \in \{1, \ldots, \ell \} : \exists i_1 < i_2 \in \{1, \ldots, m \} : I_j \cap \text{supp}(y_{i_1}) \not= \emptyset, I_j \cap \text{supp}(y_{i_2}) \not= \emptyset \}
\]

For any \( j \in U \) we define

\[
s_{j,1} = \min \{ i \in \{1, \ldots, m \} : I_j \cap \text{supp}(y_i') \not= \emptyset \}
\]

and

\[
s_{j,2} = \max \{ i \in \{1, \ldots, m \} : I_j \cap \text{supp}(y_i') \not= \emptyset \}.
\]

Then

\[
\sum_{j \in U} |I_j^* (\sum_{i=1}^{m} \alpha_i y_i')|^2 = \sum_{j \in U} |I_j (\alpha_{s_{j,1}} y_{s_{j,1}}') + I_j (\alpha_{s_{j,2}} y_{s_{j,2}}')|^2
\]

\[
\leq \sum_{j \in U} (2 \alpha_{s_{j,1}}^2 |I_j(y_{s_{j,1}}')|^2 + 2 \alpha_{s_{j,2}}^2 |I_j(y_{s_{j,2}}')|^2) \leq 4 + \frac{\epsilon^2}{32}
\]

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Without loss of generality, we may assume $F, U$ partition $\{1, ..., \ell\}$, so adding, obtain
\[
\| \sum_{i=1}^{m} \alpha_i y_i' \| < \sqrt{5 + \frac{\epsilon^2}{16}} < \sqrt{5} + \frac{\epsilon}{4}
\]
Then
\[
\| \sum_{i=1}^{m} \alpha_i u_i' \| \leq \sum_{i=1}^{m} |\alpha_i| \| u_i' - y_i' \| + \| \sum_{i=1}^{m} \alpha_i y_i' \|
\leq (\sum_{i=1}^{m} \| u_i' - y_i' \|^2)^{1/2} + \sqrt{5} + \frac{\epsilon}{4}
\leq \sqrt{5} + \frac{\epsilon}{2}
\]

Theorem 2.4.1. $J$ is $\ell_2$-saturated. In particular, for any infinite dimensional $Y$ of $J$, there is sequence $\{x_n\}_{n \in \mathbb{N}}$ in $Y$ such that for any $\epsilon > 0$ there exists subsequence $\{x_n'\}_{n \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ such that for any $m \in \mathbb{N}$ and $\alpha_1, ..., \alpha_m \in \mathbb{R}$, one has

\[
(1 - \epsilon)(\sum_{i=1}^{m} \alpha_i^2)^{1/2} \leq \| \sum_{i=1}^{m} \alpha_i x_i' \| \leq (\sqrt{5} + \epsilon)(\sum_{i=1}^{m} \alpha_i^2)^{1/2}
\]

Proof. $J$ does not contain $\ell_1$, hence Proposition 1.3.3 implies that there is sequence $\{x_n\}_{n \in \mathbb{N}}$ in $J$ which is unitary and weakly null. With a sliding hump argument, we may construct a unitary block $\{u_n\}_{n \in \mathbb{N}}$ and subsequence $\{x_n'\}_{n \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ such that

\[
\sum_{n=1}^{\infty} \| x_n' - u_n \|^2 < \frac{\epsilon^2}{4}
\]

Considering $m \in \mathbb{N}$ and $\alpha_1, ..., \alpha_m \in \mathbb{R}$ with $\sum_{i=1}^{m} \alpha_i^2 = 1$, Lemma 2.4.1 implies

\[
\| \sum_{i=1}^{m} \alpha_i x_i' \| \leq \sum_{i=1}^{m} |\alpha_i| \| x_i' - u_i \| + \| \sum_{i=1}^{m} \alpha_i u_i \| \leq \sqrt{5} + \epsilon
\]
and

\[ \left\| \sum_{i=1}^{m} \alpha_i u_i \right\| - \sum_{i=1}^{m} \left| \alpha_i \right| \left\| x_i' - u_i \right\| \leq \left\| \sum_{i=1}^{m} \alpha_i x_i' \right\| \Rightarrow 1 - \epsilon \leq \left\| \sum_{i=1}^{m} \alpha_i x_i' \right\| \]

The claim comes from Remark 1.3.1.\qed
Chapter 3

The James Tree Space

3.1 Definition of $\mathcal{JT}$ and elementary properties

Let us begin with some basic definitions on the Cantor tree, which be the natural space to define our basis. These definitions will turn out to be very important and will be used with reference in the following. We then define $\mathcal{JT}$ and prove some of its basic properties. Some of them will be properties that $\mathcal{J}$ has as well, so we will refer to Chapter 2 quite often.

- We define
  \[ 2^{<\mathbb{N}} = \{ \sigma = (\sigma_1, ..., \sigma_n) : \sigma_i \in \{0, 1\} \forall i = 1, ..., n, n \in \mathbb{N} \} \cup \{\emptyset\}. \]

  This set is called binary Cantor tree. The empty set $\emptyset$ is called root of the tree. The elements of $2^{<\mathbb{N}}$ are called nodes of the tree.

- We also define the set of sequences of 0 and 1 as follows
  \[ 2^{\mathbb{N}} = \{ \sigma : \mathbb{N} \to \{0, 1\} \}. \]

- We define the level function $|\cdot| : 2^{<\mathbb{N}} \to \mathbb{N} \cup \{0\}$ by
  \[ |s| = \begin{cases} 0 & \text{if } s = \emptyset \\ n & \text{if } s = (s_1, ..., s_n) \end{cases} \]
• We define the partial ordering '⊆' on $2^{<\mathbb{N}}$ as follows

$$\emptyset \subseteq s \quad \forall s \in 2^{<\mathbb{N}}$$

If $s, u \in 2^{<\mathbb{N}}$ with $s, u \neq \emptyset$, then $s \subseteq t \iff |s| \leq |t|$,

$$s_i = t_i \quad \forall i = 1, \ldots, |s|$$

• For $\sigma \in 2^\mathbb{N}$ and $n \in \mathbb{N}$, we will denote $\sigma|_n = (\sigma_1, \ldots, \sigma_n) \in 2^{<\mathbb{N}}$. If we consider pairwise disjoint $\sigma_1, \ldots, \sigma_n \in 2^\mathbb{N}$, then there is $N \in \mathbb{N}$ such that $\sigma_i|_N \neq \sigma_j|_N \quad \forall i, j \in \{1, \ldots, n\}$ with $i \neq j$. The minimum number with this property is called separation level of $\sigma_1, \ldots, \sigma_n$.

• Let $I \subseteq 2^{<\mathbb{N}}$ such that for any $s, t \in I$ we have either $s \subseteq t$ or $t \subseteq s$. Moreover if for any $s, t \in I$ and $w \in 2^{<\mathbb{N}}$ such that $s \subseteq w \subseteq t$, we have that $w \in I$, then $I$ is called an interval.

• Every finite interval $I$ with $n$ nodes can be written as $I = \{s_1, s_2, \ldots, s_n\}$ where $s_1 \subseteq s_2 \subseteq \ldots \subseteq s_n$. Node $s_1$ is called initial node $I$, while $s_n$ is called ending node of $I$. The initial node of $I$ will be denoted as $in(I)$, while the ending point will be denoted as $end(I)$. The nodes $in(S)$ and $end(S)$ will be called endpoints of $S$. It is clear that for any $s, t \in I$ with $s \subseteq t$, there is finite interval $S \subseteq I$ with $in(S) = s$ and $end(S) = t$. Moreover, if $S, I$ are two intervals with $|in(S)| \leq |in(I)|$ and $end(S) = end(I)$, then $I \subseteq S$. Finite intervals will be called bounded as well.

• For any infinite interval $I$ there is unique $\sigma(I) \in 2^\mathbb{N}$ and unique $n \in \mathbb{N}$ such that $I = \{\sigma(I)|_{n+k} : k = 0, 1, \ldots\}$. The interval $\sigma(I)|_n$ is called initial branch of $I$ and will be denoted by $in(I)$. This notation will be used a

• Let $I_1, \ldots, I_n$ infinite disjoint intervals. Clearly $\sigma(I_1), \ldots, \sigma(I_n)$ are both pairwise disjoint. We define the separation level of $I_1, \ldots, I_n$ as the minimum positive integer $N$ such that $\sigma(I_i)|_N \neq \sigma(I_j)|_N \quad \forall i, j \in \{1, \ldots, n\}$ with $i \neq j$ with $N \geq in(I_i) \quad \forall i = 1, \ldots, n$.

• We define $t_1 = \emptyset$ and $t_2 = (0)$. For $n > 2$ we consider $t_n = (\sigma_1, \ldots, \sigma_m)$. If for all $i = 1, \ldots, m$ we have $\sigma_i = 1$, we define $t_{n+1} = (\sigma'_1, \ldots, \sigma'_m, \sigma'_{m+1})$ where $\sigma'_i = 0$ for all $i = 1, \ldots, m+1$. Alternatively we consider $i_0 = \max\{i \in \{1, \ldots, m\} : \sigma_i = 0\}$ and define $t_{n+1} = (\sigma'_1, \ldots, \sigma'_m)$ with $\sigma'_i = \sigma_i \quad \forall i = 1, \ldots, i_0 - 1, \sigma'_{i_0} = 1$ and $\sigma'_i = 0 \quad \forall i = i_0 + 1, \ldots, m$. By
this enumeration, it is clear that \(2^{\aleph_0} = \{t_n : n \in \mathbb{N}\}\) and that \(2^{\aleph_0}\) is countable.

- We observe that for any \(n \in \mathbb{N}\), we have that \(|t_n| = \lfloor \log_2 n \rfloor\) where \([\cdot]\) denotes the integer part of a positive number.

- For \(s \in 2^{\aleph_0}\), we denote \(e_s = \mathcal{X}_s\). In particular we denote \(e_n = \mathcal{X}_{(t_n)}\). Clearly the sequences \(\{e_s\}_{s \in 2^{\aleph_0}}\) and \(\{e_n\}_{n \in \mathbb{N}}\) coincide and the represent the sequence of the characteristic functions of the nodes of the Cantor tree.

We are now in the position to define the James Tree Space.

**Definition 3.1.1.** We define

\[
\mathcal{J}T = \left\{ x : 2^{\aleph_0} \rightarrow \mathbb{R} : \sup \left\{ \sum_{i=1}^{m} \left| \sum_{t \in I_i} x(t) \right|^2 \right\} < \infty \right\}
\]

where \(\sup\) is taken over all finite families of pairwise disjoint and bounded intervals \(\{I_i\}_{i=1}^{m}\). It is immediate that \(\mathcal{J}T\) is infinite dimensional vector space. For \(x \in \mathcal{J}T\) we define

\[
||x|| = \sup \left\{ \sum_{i=1}^{m} \left| \sum_{t \in I_i} x(t) \right|^2 \right\}^{1/2}
\]

, where \(\sup\) is taken over all finite families of pairwise disjoint and bounded intervals \(\{I_i\}_{i=1}^{m}\).

**Proposition 3.1.1.** \((\mathcal{J}T, || \cdot ||)\) is a Banach space.

**Proof.** The proof is quite similar to the proof of Proposition 2.1.1. We will give it for completeness. We first show that \(|| \cdot ||\) is a norm. The only non-trivial part is triangular inequality. Consider \(x, y \in \mathcal{J}T\) and \(\{I_i\}_{i=1}^{m}\) finite
and pairwise disjoint intervals. Then Minkowski’s inequality implies

\[
\left( \sum_{i=1}^{m} \left| \sum_{t \in I_i} (x(t) + y(t)) \right| \right)^{1/2} = \left[ \left( \sum_{t \in I_1} x(t) + \sum_{t \in I_1} y(t) \right)^2 + \ldots + \left( \sum_{t \in I_m} x(t) + \sum_{t \in I_m} y(t) \right)^2 \right]^{1/2} \\
\leq \left( \sum_{i=1}^{m} \left| \sum_{t \in I_i} x(t) \right|^2 \right)^{1/2} + \left( \sum_{i=1}^{m} \left| \sum_{t \in I_i} y(t) \right|^2 \right)^{1/2} \\
\leq \|x\| + \|y\|,
\]

so

\[||x + y|| \leq ||x|| + ||y||.\]

We also show completeness. Let \{x_n\}_{n \in \mathbb{N}} be a Cauchy sequence in \( JT \). Then for any \( \epsilon > 0 \), there is \( N \) such that for any \( m > n \geq N \), we have \( ||x_m - x_n|| < \epsilon \). Considering \( t \in 2^{<\mathbb{N}} \) and the interval \( I_t = \{t\} \), the definition of the the norm implies that \( |x_m(t) - x_n(t)| < \epsilon \ \forall m > n > N \). Thus the sequence \{x_n(t)\}_{n \in \mathbb{N}} converges for any \( t \in 2^{<\mathbb{N}} \). We consider the map \( x : 2^{<\mathbb{N}} \to \mathbb{R} \), given by \( x(t) = \lim_{n \to \infty} x_n(t) \). We will show \( x \in JT \) and that \( x = \lim_{n \to \infty} x_n \). Indeed, consider \( \epsilon > 0 \) and \( M \in \mathbb{N} \) such that for any \( m > n \geq M \), we have \( ||x_m - x_n|| < \frac{\epsilon}{2} \). Then for any finite family of pairwise disjoint intervals \{I_i\}_{i=1}^{k}, we obtain

\[
\left( \sum_{i=1}^{k} \left| \sum_{t \in I_i} [x_m(t) - x_n(t)] \right|^2 \right)^{1/2} < \frac{\epsilon}{2} \ \forall m > n \geq M \Rightarrow (3.1) \\
\left( \sum_{i=1}^{k} \left| \sum_{t \in I_i} [x_n(t) - x] \right|^2 \right)^{1/2} \leq \frac{\epsilon}{2} < \epsilon \ \forall n \geq M \Rightarrow (3.2)
\]

Setting \( n = M \) in (3.1), we get that \( x_M - x \in JT \Rightarrow x \in JT \), hence \( x = \lim_{n \to \infty} x_n \). \( \square \)

**Remark 3.1.1.** Let \( x \in JT \). Consider \( k \in \mathbb{N} \) and \( s_k = \sum_{i=1}^{k} x_i e_i \). the the following estimate holds:

\[||s_k||^2 + ||x - s_k||^2 \leq ||x||^2\]

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Proof. The proof is similar to the proof of Remark 2.1.1. □

**Proposition 3.1.2.** The sequence \( \{ e_n \}_{n \in \mathbb{N}} \) is monotone and unitary Schauder basis of \( J\mathcal{T} \).

**Proof.** Clearly \( e_n \neq 0 \ \forall n \in \mathbb{N} \) and \( ||e_n|| = 1 \ \forall n \in \mathbb{N} \). We first show that \( J\mathcal{T} = [e_n : n \in \mathbb{N}] \). Consider \( x \in J\mathcal{T} \). Let us define \( s_n = \sum_{i=1}^{n} x(t_i)e_i \).

We will show that \( x = \lim_{n \to \infty} s_n \). Indeed, let \( \epsilon > 0 \). The definition of the norm yields that there are pairwise disjoint and bounded intervals \( \{ I_i \}_{i=1}^{m} \) such that,

\[
\sum_{i=1}^{m} \left| \sum_{t \in I_i} x(t) \right|^2 > ||x||^2 - \epsilon^2
\]  
(3.3)

For \( n_0 = \max \left\{ |t| : t \in \bigcup_{i=1}^{m} I_i \right\} \) (3.3) yields that

\[
||x||^2 - ||s_n||^2 < \epsilon^2 \ \forall n \geq 2^{n_0+1}
\]

So for \( n \geq 2^{n_0+1} \), last Remark implies that \( ||x - s_n||^2 = ||x||^2 - ||s_n||^2 - ||s_n||^2 = ||x||^2 - ||s_n||^2 < \epsilon^2 \). So \( x = \lim_{n \to \infty} s_n \). Finally for \( n \in \mathbb{N} \) and \( \alpha_1, ..., \alpha_n, \alpha_{n+1} \in \mathbb{R} \), we consider pairwise disjoint and bounded intervals \( \{ I_i \}_{i=1}^{m} \) with \( t_{n+1} \notin \bigcup_{i=1}^{m} I_i \). Then

\[
(\sum_{i=1}^{m} | \sum_{t_j \in I_i} \alpha_j |^2)^{1/2} \leq || \sum_{i=1}^{n+1} \alpha_i e_i || \Rightarrow || \sum_{i=1}^{n} \alpha_i e_i || \leq || \sum_{i=1}^{n+1} \alpha_i e_i ||
\]

and the claim is proved. □

**Proposition 3.1.3.** the basis \( \{ e_n \}_{n \in \mathbb{N}} \) is boundedly complete. Therefore \( c_0 \) does not embed in \( J\mathcal{T} \).

**Proof.** The proof is similar to Proposition 2.1.3. □

**Remark 3.1.2.** \( J\mathcal{T} \) can be equivalently defined as the completion of \( < c_{00}(2^{\mathbb{N}}) > \) under the norm defined.
Remark 3.1.3. Proof. In analogy to Remark 2.1.3, if \( I \) is a bounded interval, we define \( I^* \equiv \sum_{s \in I} e^*_s \in \mathcal{J}T^* \). Similarly, for \( I \) infinite interval, we define \( I^* \equiv \sum_{s \in I} e^*_s \in \mathcal{J}T^* \). Clearly \( ||I^*|| = 1 \) for any interval \( I \). In particular, for \( \sigma \in 2^\mathbb{N} \), we will denote \( \sigma^* \equiv \sum_{n=1}^{\infty} e^*_{\sigma|n} \). We have that \( I \not\in [e^*_n : n \in \mathbb{N}] \) for any infinite interval \( I \), therefore the basis \( \{e_n\}_{n \in \mathbb{N}} \) is not shrinking, hence \( \mathcal{J}T \) is not reflexive. Finally, for any \( x \in \mathcal{J}T \), we have the following norm description

\[
|||x||| = \sup \left\{ \sum_{i=1}^{m} |I^*_i(x)|^2 \right\}^{1/2},
\]

where \( \{I_i\}_{i=1}^{m} \) are pairwise disjoint, maybe infinite, intervals. \( \square \)

Proposition 3.1.4. For any \( \sigma \in 2^\mathbb{N} \), the subspace \( [e^*_{\sigma|n}, n \in \mathbb{N}] \) is isometric to \( \mathcal{J} \).

Proof. Let \( T : < e^*_{\sigma|n}, n \in \mathbb{N} > \rightarrow \mathcal{J} \) given by

\[
T(\sum_{i=1}^{n} \lambda_i e_{\sigma|i}) = (\lambda_1, ..., \lambda_n, 0, ...).
\]

The map \( T \) is clearly linear. Considering pairwise disjoint intervals of \( \mathbb{N} \), \( I_1 = [n_1, n'_1], ..., I_m = [n_m, n'_m] \) and the pairwise disjoint intervals, \( S_1 = [\sigma|n_1, \sigma|n'_1], ..., S_m = [\sigma|n_m, \sigma|n'_m] \cap 2^\mathbb{N} \), we obtain

\[
\sum_{j=1}^{m} |I^*_j(\sum_{i=1}^{n} \lambda_i e_i)|^2 = \sum_{j=1}^{m} |S^*_j(\sum_{i=1}^{n} \lambda_i e_{\sigma|i})|^2 \Rightarrow ||T(x)|| = ||x||
\]

Hence, \( T \) is an isometry which can be extended by density on \( [e^*_{\sigma|n}, n \in \mathbb{N}] \).

It remains to show it is surjective as well. Indeed, let \( x = \sum_{n=1}^{\infty} \lambda_n e_n \in \mathcal{J} \).

Then for any \( \epsilon > 0 \), there is \( N \in \mathbb{N} \) such that for any \( m > n > N \), we have

\[
|| \sum_{i=n+1}^{m} \lambda_i e_{\sigma|i} || = || \sum_{i=n+1}^{m} \lambda_i e_i || < \epsilon.
\]

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Therefore the series \( \sum_{n=1}^{\infty} \lambda_n e_{\sigma_n} \) converges and \( T(\sum_{n=1}^{\infty} \lambda_n e_{\sigma_n}) = \sum_{n=1}^{\infty} \lambda_n e_n \). The claim is proved.

**Proposition 3.1.5.** The conjugate space \( J^* T \) is not separable.

**Proof.** The set \( \{ \sigma^* : \sigma \in 2^N \} \) is clearly uncountable and 1-separated. Indeed, considering \( \sigma_1^* \neq \sigma_2^* \), there is \( s \in 2^{<N} \) with \( \sigma_1^*(e_s) = 1 \) and \( \sigma_2^*(e_s) = 0 \). Thus \( ||\sigma_1^* - \sigma_2^*|| \geq 1 \) so \( J^* T \) is not separable.

**Proposition 3.1.6.** For any \( \sigma \in 2^N \), there is \( \sigma^{**} \in J^{**} T \) with \( \hat{e}_{\sigma_n} \xrightarrow{w^*} \sigma^{**} \).

**Proof.** We have shown there is \( T : [e_{\sigma_n} : n \in N] \to J \) surjective isometry. Therefore the conjugate operator \( T^* : J^* \to [e_{\sigma_n} : n \in N]^* \) is surjective isometry. Since \( J^* = [e_n^* : n \in N] \oplus [s^*] \), we have that for any \( x^* \in [e_{\sigma_n} : n \in N]^* \) there is unique sequence of real numbers \( (\lambda_n)_{n \in N} \) and unique \( \lambda \in \mathbb{R} \), such that

\[
x^* = T^* \left( \sum_{n=1}^{\infty} \lambda_n e_n^* + \lambda s^* \right) = \sum_{n=1}^{\infty} \lambda_n (e_n^* \circ T) + \lambda (s^* \circ T)
\]

We show \( (e_{\sigma_k})_{k \in N} \) is \( w \)-Cauchy. Indeed, if \( x^* \in J^* T \) then \( x^* \bigg|_{[e_{\sigma_n} : n \in N]} \in [e_{\sigma_n} : n \in N]^* \) and so for any \( k \in N \), we get

\[
x^*(e_{\sigma_k}) = \sum_{n=1}^{\infty} \lambda_n e_n^* (T(e_{\sigma_k})) + \lambda s^* (T(e_{\sigma_k})) = \sum_{n=1}^{\infty} \lambda_n e_n^*(e_k) + \lambda s^*(e_k)
\]

\[
= \lambda_k + \lambda \xrightarrow{k \to \infty} \lambda,
\]

since \( \lambda_k \xrightarrow{k \to \infty} 0 \). Equivalently, there is \( \sigma^{**} \in J^{**} T \) with \( \sigma^{**} = w^* - \lim_{k \to \infty} \hat{e}_{\sigma_k} \). It remains to show \( \sigma^{**} \notin J^* T \). We have that \( \sigma^{**}(\sigma^*) = \lim_{n \to \infty} \sigma^*(e_{\sigma_n}) = 1 \).

If we assume that \( \sigma^{**} \in J^* T \), then \( \sigma^{**} \) is \( w^* \)-continuous, hence \( \sigma^{**}(\sigma^*) = \sum_{n=1}^{\infty} \sigma^{**}(e_{\sigma_n}^*) = 0 \) which leads to contradiction. The result is proved.
3.2 $\ell_1$ does not embed in $J\mathcal{T}$

The goal of this section is to show that $\ell_1$ does not embed in $J\mathcal{T}$. Compared to the case of $J$ where this conclusion is immediate, additional arguments are needed, since $J\mathcal{T}^*$ is not separable.

Let $\{I_n\}_{n \in \mathbb{N}}$ pairwise disjoint intervals of $2^{<\mathbb{N}}$. For any $x \in J\mathcal{T}$ we have that

$$\sum_{n=1}^{\infty} |I_n^*(x)|^2 \leq ||x||^2.$$ 

Moreover, for any sequence $\{\alpha_n\}_{n \in \mathbb{N}} \in B_{\ell_2}$, Cauchy-Schwartz implies that

$$\sum_{n=1}^{\infty} |\alpha_n||I_n^*(x)| \leq ||x||,$$

so $\sum_{n=1}^{\infty} \alpha_n I_n^*(x)$ converges absolutely for any $x \in J\mathcal{T}$ and $w^* - \sum_{n=1}^{\infty} \alpha_n I_n^* \in B_{J\mathcal{T}^*}$. Let us define

$$K = \left\{ w^* - \sum_{n=1}^{\infty} \alpha_n I_n^* : \{\alpha_n\}_{n \in \mathbb{N}} \in B_{\ell_2} \text{ and } \{I_n\}_{n \in \mathbb{N}} \text{ pairwise disjoint intervals} \right\}$$

Clearly $K \subseteq B_{J\mathcal{T}^*}$.

**Proposition 3.2.1.** The set $K$ is norming for $J\mathcal{T}$ i.e.

$$||x|| = \sup \{k^*(x) : k^* \in K \} \quad \forall x \in J\mathcal{T}.$$ 

We may also write $||x|| = \sup \{|k^*(x)| : k^* \in K\}$.

**Proof.** Let $x \in J\mathcal{T}$. Since $K \subseteq B_{J\mathcal{T}^*}$, we have that

$$\sup \{k^*(x) : k^* \in K \} \leq ||x||$$

For the opposite direction, consider $n \in \mathbb{N}$ and define $x_n = \sum_{i=1}^{n} e_i^*(x)e_i$. Then there are pairwise disjoint intervals $\{I_i\}_{i=1}^{m}$ such that

$$||x_n|| - \frac{1}{n} < \left(\sum_{i=1}^{m} |I_i^*(x_n)|^2\right)^{1/2}.$$
Defining \( \lambda_i = \frac{I_i^*(x_n)}{(\sum_{i=1}^{m} |I_i^*(x_n)|^2)^{1/2}} \), we have that \( k^* = \sum_{i=1}^{m} \lambda_i I_i^* \in K \) and

\[
k^*(x_n) = \left( \sum_{i=1}^{m} |I_i^*(x_n)|^2 \right)^{1/2},
\]

so \( ||x_n|| - \frac{1}{n} < k^*(x_n) \leq \sup \{k^*(x_n) : k^* \in K\} \) \( \overset{n \to \infty}{\to} \) \( ||x|| \leq \sup \{k^*(x) : k^* \in K\} \) and the result follows. The second description is immediate.

**Proposition 3.2.2.** The set \( K \) is \( w^* \)-compact subset of \( JT^* \).

**Proof.** Since \( JT \) is separable, the ball \((B_{X^*}, w^*)\) is a metric space, so it suffices to show that \( K \) is sequentially compact. Let \( k_n^* = w^* - \sum_{i=1}^{\infty} \alpha_{i,n} I_{i,n}^* \) a sequence in \( K \). Since this series converges absolutely, we may assume, after a possible re-ordering, that for any \( n \in \mathbb{N} \), we have that \( |\alpha_{i,n}| \leq |\alpha_{i,n}| \) \( \forall i \in \mathbb{N} \). We first prove a claim.

- **Claim:** Let \( \{I_n\}_{n \in \mathbb{N}} \) a sequence of intervals. Then there is subsequence \( \{I_{k_n}\}_{n \in \mathbb{N}} \) and interval \( I \) such that \( I_{k_n} \overset{w^*}{\to} I^* \).

**Proof of the claim** Using a diagonal argument, we may find subsequence \( \{I_{k_n}\}_{n \in \mathbb{N}} \) such that \( \{I_{k_n}(e_s)\}_{n \in \mathbb{N}} \) converges for any \( s \in 2^{<\mathbb{N}} \). Let us define \( I = \{s \in 2^{<\mathbb{N}} : \exists n_s \in \mathbb{N} \text{ with } s \in I_{k_n} \forall n \geq n_s\} \). Then \( I \) is an interval. Indeed, consider \( s, t \in I \). Then there is \( n_0 \in \mathbb{N} \) such that \( s, t \in I_{k_{n_0}} \). Thus, either \( s \subseteq t \) or \( t \subseteq s \). Let us consider \( s, t \in I \) and \( w \in 2^{<\mathbb{N}} \) such that \( s \subseteq w \subseteq t \). From the definition of \( I \), there is \( n_0 \in \mathbb{N} \) such that \( s, t \in I_n \forall n \geq n_0 \). Hence, we have that \( w \in I_n \forall n \geq n_0 \Rightarrow w \in I \). We may easily show that \( \lim_{n \to \infty} I_{k_n}^*(s) = I^*(s) \forall s \in 2^{<\mathbb{N}} \) and since \( ||I_{k_n}^*|| = 1 \forall n \in \mathbb{N} \), Corollary 1.1.1, implies the claim.

**Main proof** Using a diagonal argument, we may find \( M \in [\mathbb{N}] \), sequence \( \{\alpha_i\}_{i \in \mathbb{N}} \in B_{\ell_2} \) such that \( \alpha_{i,n} \overset{n \in M}{\to} \alpha_i \forall i \in \mathbb{N} \) and intervals \( \{I_i\}_{i \in \mathbb{N}} \) such that \( I_i = w^* - \lim_{n \in M} I_{i,n}^* \). The intervals \( \{I_i\}_{i \in \mathbb{N}} \) are clearly pairwise disjoint. We define \( k^* = w^* - \sum_{n=1}^{\infty} \alpha_i I_{i,n}^* \in K \) and we will show that \( k^* = w^* - \lim_{n \in M} k_n^* \). Indeed, consider \( s \in 2^{<\mathbb{N}} \) and \( \epsilon > 0 \). We pick \( N \in \mathbb{N} \) such that

\[
(\sum_{i=N+1}^{\infty} \alpha_i^2)^{1/2} < \frac{\epsilon}{4}.
\]

Then there is \( n_0 \in \mathbb{N} \) such that for any \( n \geq n_0 \), we have
that
\[ \sum_{i=1}^{N} |\alpha_{i,n}I_{i,n}^*(e_s) - \alpha_i I_i^*(e_s)| < \frac{\epsilon}{4} \]
and
\[ |\alpha_{N+1,n} - \alpha_{N+1}| < \frac{\epsilon}{4}. \]
Then for any \( n \in M \) with \( n \geq n_0 \), we have
\[
\left| \sum_{i=1}^{\infty} \alpha_{i,n}I_{i,n}^*(e_s) - \sum_{i=1}^{\infty} \alpha_i I_i^*(e_s) \right| \leq \sum_{i=1}^{N} |\alpha_{i,n}I_{i,n}^*(e_s) - \alpha_i I_i^*(e_s)|
\]
\[ + \left| \sum_{i=N+1}^{\infty} \alpha_{i,n}I_{i,n}^*(e_s) \right| + \left| \sum_{i=N+1}^{\infty} \alpha_i I_i^*(e_s) \right| \]
\[ < \frac{\epsilon}{4} + \left( \sum_{i=N+1}^{\infty} \alpha_i^2 \right)^{1/2} \left( \sum_{i=N+1}^{\infty} |I_i^*(e_s)|^2 \right)^{1/2}
\]
\[ + |\alpha_{j,n}|, \quad \text{for some} \quad j \geq N + 1 \]
\[ < \frac{\epsilon}{2} + \frac{\epsilon}{4} + |\alpha_{N+1,n}| \]
\[ \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + |\alpha_{N+1} - \alpha_{N+1}| + |\alpha_{N+1}| \]
\[ \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon. \]

Therefore \( k_n^*(e_s) \xrightarrow{n \in M} k^*(e_s) \quad \forall s \in \mathbb{2}^{<\mathbb{N}} \) and the result is proved.

We will now need to use some measure theoretic arguments. Let \((X, \mathcal{A}, \mu)\) a signed measure space and \( f \in L^1(|\mu|) \). The integral of \( f \) with respect to \( \mu \) is defined as
\[
\int_X f \, d\mu = \int_X f \, d\mu^+ - \int_X f \, d\mu^-.
\]
Dominated Convergence Theorem implies that if we consider a sequence of measurable functions \( f_n : (X, \mathcal{A}) \to \mathbb{R} \) with \( f = \lim_{n \to \infty} f_n \) such that there is \( g \in L^1(|\mu|) \) with \( |f_n| \leq g \quad \forall n \in \mathbb{N} \) a.e. in \( X \) then
\[
\lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0 \Rightarrow \lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu
\]
We will now use a special form of Riesz Representation Theorem (see [4]). Let \( X \) be a topological space. A signed Borel measure \( \mu \) on \( X \) is called regular if \( |\mu| \) is a regular positive measure. We will write \( \mathcal{M}_r(X) \) for the set of all regular, signed Borel measures of \( X \).
Theorem 3.2.1. (Riesz’s Representation Theorem) Let $X$ be a compact Hausdorff space. Then for any $f^* \in C^*(X)$ there is unique $\mu \in M_r(X)$ such that $f^*(f) = \int_X f \, d\mu \ \forall f \in C(X)$.

In the following, we will write $C(K)$ for all real $w^*$-continuous functions on $K$ under the $\| \cdot \|_\infty$ norm.

Proposition 3.2.3. Let $\{x_n\}_{n \in \mathbb{N}}$ be a bounded sequence in $JT$. If the sequence $\{I^*(x_n)\}_{n \in \mathbb{N}}$ converges for any interval $I$ then $\{x_n\}_{n \in \mathbb{N}}$ is $w$-Cauchy.

Proof. Let $M = \sup_{n \in \mathbb{N}} \{||x_n||\}$. We define $T : JT \to C(K)$ by $T(x) = \hat{x}|_K$. Clearly $T$ is a well-defined linear isometry. Indeed, $||T(x)||_\infty = ||\hat{x}|_K||_\infty = \sup \{|\hat{x}(x^*)| : x^* \in K\} = \sup \{|x^*(x)| : x^* \in K\} = ||x||$, since $K$ is norming of $JT$. Therefore the conjugate map $T^* : C^*(K) \to JT^*$ is surjective. So, for any $x^* \in JT^*$, there is $f^* \in C^*(K)$ such that $x^* = f^* \circ T$. Riesz’s Representation Theorem implies that for any $x^* \in JT^*$, there is unique $\mu_{x^*} \in M_r(K)$ such that for any $n \in \mathbb{N}$, we have $x^*(x_n) = \int_K \hat{x}_n|_K, d\mu_{x^*}$. Since $||x_n||_K| \leq M$ for any $n \in \mathbb{N}$ and $|\mu_{x^*}|(K) < \infty$ for any $x^* \in JT^*$, Dominated Convergence Theorem yields it is enough to show that $\{x^*(x_n)\}_{n \in \mathbb{N}}$ converges for all $x^* \in K$. Indeed, let $x^* = w^* - \sum_{i=1}^{\infty} \lambda_i I^*_i \in K$. By assumption we may write $\alpha_i = \lim_{n \to \infty} I^*_i(x_n)$. Then $\left( \sum_{i=1}^{\infty} \alpha_i^2 \right)^{1/2} \leq M$ and $\sum_{i=1}^{\infty} |\lambda_i \alpha_i| \leq M$, by Cauchy-Schwartz inequality. Hence, $\sum_{i=1}^{\infty} \lambda_i \alpha_i$ is absolutely convergent. Consider $\epsilon > 0$ and $N \in \mathbb{N}$ such that $\left( \sum_{i=N+1}^{\infty} \lambda_i^2 \right)^{1/2} < \frac{\epsilon}{3M}$ and $|\sum_{i=N+1}^{\infty} \lambda_i \alpha_i| < \frac{\epsilon}{3}$. Fixing $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ there holds $\sum_{i=1}^{N} |\lambda_i I^*_i(x_n) - \lambda_i \alpha_i| < \frac{\epsilon}{3}$.
for \( n \geq n_0 \) we obtain

\[
| x^* (x_n) - \sum_{i=1}^{\infty} \lambda_i \alpha_i | \leq \sum_{i=1}^{N} | \lambda_i I_i^* (x_n) - \lambda_i \alpha_i | + \sum_{i=N+1}^{\infty} | \lambda_i I_i^* (x_n) | + \sum_{i=N+1}^{\infty} | \lambda_i \alpha_i |
\]

\[
< \frac{\epsilon}{3} + ( \sum_{i=N+1}^{\infty} \lambda_i^2 )^{1/2} ( \sum_{i=N+1}^{\infty} | I_i^* (x_n) |^2 )^{1/2} + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

We will use the following notation. Given a countable set \( M \), we will write \([M]\) to denote the set of infinite subsets of \( M \). Let us first prove the following Lemma.

**Lemma 3.2.1.** Let \( X \neq \emptyset \) and \( f_n : X \to \mathbb{R} \) sequence of functions such that

\[
\sup_{n \in \mathbb{N}} \{ |f_n(x)| \} < \infty \quad \forall x \in X.
\]

If for any \( \epsilon > 0 \) and \( M \in \mathbb{N} \) there is \( L \in [M] \) with

\[
\limsup_{n \in L} f_n(x) - \liminf_{n \in L} f_n(x) < \epsilon, \quad \forall x \in X
\]

then the sequence \( \{f_n\}_{n \in \mathbb{N}} \) has pointwise subsequence.

**Proof.** By induction we will construct a decreasing sequence \( \{L_k\}_{k \in \mathbb{N}} \) of infinite subsets of \( \mathbb{N} \) such that for any \( k \in \mathbb{N} \) we have

\[
\limsup_{n \in L_k} f_n(x) - \liminf_{n \in L_k} f_n(x) < \frac{1}{k}, \quad \forall x \in X.
\]

Consider a strictly increasing sequence \( \{n_k\}_{k \in \mathbb{N}} \) such that \( n_k \in L_k \quad \forall k \in \mathbb{N} \) and the set \( L_\infty = \{ n_k : k \in \mathbb{N} \} \). Clearly \( L_\infty \) is infinite and for any \( k \in \mathbb{N} \) we have \( L_\infty \subseteq L_k \). Then for all \( x \in X \) we have that

\[
\limsup_{n \in L_\infty} f_n(x) - \liminf_{n \in L_\infty} f_n(x) \leq \limsup_{n \in L_k} f_n(x) - \liminf_{n \in L_k} f_n(x) < \frac{1}{k}, \quad \forall k \in \mathbb{N},
\]

so letting \( k \to \infty \), we obtain

\[
\limsup_{n \in L_\infty} f_n(x) = \liminf_{n \in L_\infty} f_n(x)
\]

thus the sequence \( \{f_n\}_{n \in L_\infty} \) is pointwise convergent. \( \square \)
We are now able to prove the non-embedding of $\ell_1$ in $\mathcal{JT}$.

**Theorem 3.2.2.** $\ell_1$ does not embed in $\mathcal{JT}$.

*Proof.* Assume $\ell_1$ embeds in $\mathcal{JT}$. Then Proposition 1.4.2 implies there is block of the basis of $\mathcal{JT}$ equivalent to the standard basis of $\ell_1$. We will show that any block has $w$-Cauchy subsequence, which contradicts the $\ell_1$-Theorem. Let $\{u_n\}_{n \in \mathbb{N}}$ a block and we write $M = \sup_{n \in \mathbb{N}} \{||u_n||\}$. We will show there is subsequence $\{u_{nk}\}_{k \in \mathbb{N}}$ such that $\{I^*(u_{nk})\}_{k \in \mathbb{N}}$ converges for any interval $I$ and the contradiction will come by Proposition 3.2.3. For bounded intervals, using a diagonal argument, we may find $K \in [\mathbb{N}]$ such that $\{S^*(u_n)\}_{n \in K}$ converges for any bounded interval $S$. Hence, using Lemma 3.2.1, it suffices to show that for any $\epsilon > 0$ and $M \in [K]$, there is $L \in [M]$ such that $\limsup_{n \in L} \sigma^*(u_n) - \liminf_{n \in L} \sigma^*(u_n) \leq \epsilon$ for any $\sigma \in 2\mathbb{N}$. Arguing by contradiction, assume there is $\epsilon > 0$ and $M \in [K]$, such that for any $L \in [M]$, there is $\sigma_L \in 2\mathbb{N}$ with

$$\limsup_{n \in L} \sigma^*_L(u_n) - \liminf_{n \in L} \sigma^*_L(u_n) > \epsilon \Rightarrow \limsup_{n \in L} \sigma^*_L(u_n) + \liminf_{n \in L} \sigma^*_L(u_n) > \frac{\epsilon^2}{2} \quad (3.4)$$

We pick $k \in \mathbb{N}$ such that $k^2 \epsilon^2 > M^2$. Consider $L_0 \in [M]$ and $\sigma_1 \in 2\mathbb{N}$ such that $(3.4)$ holds. Then at least one of $\limsup_{n \in L_0} \sigma^*_1(u_n)$ and $\liminf_{n \in L_0} \sigma^*_1(u_n)$ is grater than $\frac{\epsilon^2}{4}$. Hence, there is $L_1 \in [L_0]$ such that $\limsup_{n \in L_1} \sigma^*_1(u_n) > \frac{\epsilon^2}{4}$ and $\sigma_2 \in 2\mathbb{N}$ which satisfies $(3.4)$. It is clear that $\sigma_1 \neq \sigma_2$. Continuing inductively we may find $L_k \subseteq L_{k-1} \subseteq \ldots \subseteq L_1 \subseteq \mathbb{N}$ and $\sigma_1, \ldots, \sigma_k \in 2\mathbb{N}$ pairwise distinct such that $\limsup_{n \in L_i} \sigma^*_i(u_n) > \frac{\epsilon^2}{4}$ for any $i = 1, \ldots, k$, thus $\limsup_{n \in L_k} \sigma^*_i(u_n) > \frac{\epsilon^2}{4}$ $\forall i = 1, \ldots, k$. Therefore there is $N \in L_k$ such that for any $i = 1, \ldots, k$, we have $\sigma^*_i(u_n) > \frac{\epsilon^2}{4}$ $\forall n \in L_k : n \geq N$. Since $\sigma_1, \ldots, \sigma_k$ finally separate, since they are pairwise distinct and the sequence $\{u_n\}_{n \in \mathbb{N}}$ is block, we may find $n_0 \in L_k$ with $n_0 \geq N$ such that $||u_{n_0}||^2 \geq \sum_{i=1}^k \sigma^*_i(u_{n_0}) > k \frac{\epsilon^2}{4} > M^2$ which contradicts the boundness assumption on $\{u_n\}_{n \in \mathbb{N}}$. The proof is complete. \qed

**Corollary 3.2.1.** Every bounded sequence in $\mathcal{JT}$ has $w$–Cauchy subsequence.
Proof. It comes immediately by the fact that $\ell_1$ does not embed in $J\mathcal{T}$ and the $\ell_1$-Theorem.

Remark 3.2.1. We have seen that $J\mathcal{T}$ is not reflexive and does not contain neither $\ell_1$ nor $c_0$. Consequently, it does not have any unconditional basis.

3.3 Description of the first and second conjugate

We have seen that $J\mathcal{T}^*$ is not separable. In this section, we will give a more detailed description of the first and second conjugate spaces. Our approach will strongly rely on the following Theorem, due to R. Haydon, see [5].

Theorem 3.3.1. Let $X$ be a Banach space not containing $\ell_1$. Then for any $w^*$-compact $K \subseteq X^*$, one has $\text{conv}^w(K) = \text{conv}||\cdot||(K)$

Using this Theorem we obtain the following description for $J\mathcal{T}^*$.

Theorem 3.3.2. We have the following description for the first conjugate:

$$J\mathcal{T}^* = [I^* : I \text{ interval of } 2^{\mathbb{N}}] = [I^* : I \text{ infinite interval of } 2^{\mathbb{N}}]$$

Proof. Denoting $K$ the norming set of $J\mathcal{T}$ we used in the previous section and using Haydon’s Theorem, it suffices to show that $B_{J\mathcal{T}^*} = \text{conv}||\cdot||(K) = \text{conv}^w(K)$. We clearly have that $K \subseteq B_{J\mathcal{T}^*} \Rightarrow \text{conv}^w(K) \subseteq B_{J\mathcal{T}^*}$, since $B_{J\mathcal{T}^*}$ is $w^*$-compact. To prove the opposite direction, assume there is $x^* \in B_{J\mathcal{T}^*} \setminus \text{conv}^w(K)$. Then the Third Separating Theorem for the $w^*$-topology yields that there is $x \in J\mathcal{T}$ with

$$\sup \{ k^*(x) : k^* \in \text{conv}^w(K) \} < x^*(x) \Rightarrow \sup \{ k^*(x) : k^* \in K \} < x^*(x) \leq ||x||$$

which contradicts the fact that $K$ is norming of $J\mathcal{T}$. 

We now attempt to describe the second conjugate space. For this purpose, let us define the space

$$\ell_2(2^\mathbb{N}) = \left\{ f : 2^\mathbb{N} \to \mathbb{R} : \sup \left\{ \sum_{\sigma \in F} |f(\sigma)|^2 : F \subseteq 2^\mathbb{N} \text{ finite} \right\} < \infty \right\}$$
The mapping

\[ <f, g> = \sup \left\{ \sum_{\sigma \in F} f(\sigma)g(\sigma) : F \subseteq 2^N \text{ finite} \right\}^{1/2} \]

is easily seen to be an inner product and \( \ell_2(2^N) \) with the norm induced by this inner product is a Hilbert space. Therefore Riesz’s Theorem for Hilbert spaces implies that the mapping \( S : \ell_2(2^N) \to \ell_2^*(2^N) \), given by \( S(f)(g) = <f, g> \quad \forall g \in \ell_2(2^N) \) is linear, surjective isometry.

**Remark 3.3.1.** Let \( f \in \ell_2(2^N) \). Then there are sequences \( \{\lambda_n\}_{n \in \mathbb{N}} \in \ell_2 \) and \( \{\sigma_n\}_{n \in \mathbb{N}} \subseteq 2^N \) such that \( f = \sum_{n=1}^{\infty} \lambda_n \chi_{\{\sigma_n\}} \)

**Proof.** It is clear that \( \text{supp} f = \bigcup_{n=1}^{\infty} \left\{ \sigma \in 2^N : |f(\sigma)| > \frac{1}{n} \right\} \). We will show \( \text{supp} f \) is countable. It suffices to show that \( \left\{ \sigma \in 2^N : |f(\sigma)| > \frac{1}{n} \right\} \) is finite for any \( n \in \mathbb{N} \). Indeed, if this is not the case, there is \( n \in \mathbb{N} \) such that for any \( k \in \mathbb{N} \) we find \( \sigma_1, ..., \sigma_k \in 2^N \) such that \( |f(\sigma_i)| > \frac{1}{n} \quad \forall i = 1, ..., k \). So

\[ \frac{k}{n^2} < \sum_{i=1}^{k} |f(\sigma_i)|^2 \leq ||f||^2, \]

which is contradiction since \( k \) is arbitrary. Let us write \( \text{supp} f = \{\sigma_n : n \in \mathbb{N}\} \). Then

\[ || \sum_{i=1}^{k} f(\sigma_i)\chi_{\{\sigma_i\}} - f || = \sup_{n \in \mathbb{N}} \left\{ \sum_{i=k+1}^{n} |f(\sigma_i)|^2 \right\} = \sum_{i=k+1}^{\infty} |f(\sigma_i)|^2 \xrightarrow{k \to \infty} 0 \]

Thus \( \{f(\sigma_n)\}_{n \in \mathbb{N}} \in \ell_2 \) and \( f = \sum_{n=1}^{\infty} f(\sigma_n)\chi_{\{\sigma_n\}} \).

We will also use the following Lemma.

**Lemma 3.3.1.** Let \((X, d)\) be a metric space and an increasing sequence of subsets \((Y_m)_{m \in \mathbb{N}} \subseteq \mathcal{P}(X)\). If \( Y = \bigcup_{m=1}^{\infty} Y_m \), then

\[ \text{dist}(x, Y) = \lim_{m \to \infty} \text{dist}(x, Y_m) \quad \forall x \in X. \]
Proof. Let \( x \in X \). The sequence \((\text{dist}(x, Y_m))_{m \in \mathbb{N}}\) is non-increasing and lower bounded, hence it converges. Clearly \( Y_m \subseteq Y \ \forall m \in \mathbb{N} \Rightarrow \text{dist}(x, Y) \leq \lim_{m \to \infty} \text{dist}(x, Y_m) \). For the opposite direction, consider \( y \in Y \). Then there is a sequence \( \{y_k\}_{k \in \mathbb{N}} \subseteq \bigcup_{m=1}^{\infty} Y_m \) with \( y_k \xrightarrow{k \to \infty} y \). We inductively construct a strictly increasing sequence \( \{m_k\}_{k \in \mathbb{N}} \) of positive integers with \( y_k \in Y_{m_k} \ \forall k \in \mathbb{N} \). Indeed, let \( m_1 \in \mathbb{N} \) with \( y_1 \in Y_{m_1} \). Consider \( m_2' \in \mathbb{N} \) such that \( y_2 \in Y_{m_2'} \) and define \( m_2 = \max\{m_1, m_2'\} + 1 > m_1 \). Clearly \( y_2 \in Y_{m_2} \). Continuing inductively, given \( y_k \in Y_{m_k} \), we consider \( m_{k+1}' \in \mathbb{N} \) such that \( y_{k+1} \in Y_{m_{k+1}'} \) and we define \( m_{k+1} = \max\{m_k, m_{k+1}'\} + 1 \). It is clear that the sequence \( \{m_k\}_{k \in \mathbb{N}} \) has the required property. Then

\[
\text{dist}(x, Y_{m_k}) \leq d(x, y_k) \ \forall k \in \mathbb{N} \\
\Rightarrow \lim_{m \to \infty} \text{dist}(x, Y_m) = \lim_{k \to \infty} \text{dist}(x, Y_{m_k}) \leq \lim_{k \to \infty} d(x, y_k) = d(x, y).
\]

Thus \( \lim_{m \to \infty} \text{dist}(x, Y_m) \leq \text{dist}(x, Y) \) and the result follows.

In the following, we will denote \( Y = [e_s^* : s \in \mathbb{N}] \). As known, the quotient operator \( Q : \mathcal{JT}^* \to \mathcal{JT}^*/Y \) given by \( Q(x^*) = x^* + Y \) is surjective and bounded. Hence

\[
\mathcal{JT}^*/Y = Q[\mathcal{JT}^*] \\
= Q(<I^* : I \text{ infinite interval}>) \\
\subseteq Q[<I^* : I \text{ infinite interval}>] \\
= [I^* + Y : I \text{ infinite interval}],
\]

so \( \mathcal{JT}^*/Y = [I^* + Y : I \text{ infinite interval}] \).

**Proposition 3.3.1.** The quotient space \( \mathcal{JT}^*/Y \) is isometric to \( \ell_2(2^\mathbb{N}) \).

**Proof.** We first notice that the set \( <I^* + Y : I \text{ infinite interval} > \) is linearly independent. Moreover, considering infinite intervals \( I, S \) such that \( I^* + Y = S^* + Y \), then \( \sigma(I) = \sigma(S) \), since \( I^* - S^* \in Y \). So we may define \( U : <I^* + Y : I \text{ infinite interval} > \to \ell_2(2^\mathbb{N}) \) by

\[
U(\sum_{i=1}^{n} \lambda_i I_i^*) = \sum_{i=1}^{n} \lambda_i \mathcal{X}_{\sigma(i)}.
\]


The map $U$ is clearly well defined and linear. We show it is isometry. Without
loss of generality, we consider $I_1, \ldots, I_n$ infinite intervals $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with
$\sum_{i=1}^{n} \lambda_i^2 = 1$. It suffices to show $\| \sum_{i=1}^{n} \lambda_i I_i^* + Y \| = 1$. We get

$$\left| \sum_{i=1}^{n} \lambda_i I_i^*(x) \right| \leq \left( \sum_{i=1}^{n} \lambda_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} I_i^*(x) \right)^{1/2} \leq ||x|| \quad \forall x \in JT \Rightarrow$$

$$\Rightarrow \left\| \sum_{i=1}^{n} \lambda_i I_i^* \right\| \leq 1 \Rightarrow \left\| \sum_{i=1}^{n} \lambda_i I_i^* + Y \right\| \leq 1$$

For the other direction, let $N$ be the separation level of $I_1, \ldots, I_n$. We define
$Y_m = \langle e^*_s : |s| \leq m \rangle$. Then $Y = \bigcup_{m=1}^{\infty} Y_m$. For any $m \geq N$, we define

$$x_m = \sum_{i=1}^{n} \lambda_i e_{\sigma(I_i)|m+1}. \text{ Clearly } ||x_m|| = 1 \quad \forall m \geq N. \; \text{ Then for any } m \geq N$$

and any $y^* \in Y_m$, we get

$$\left| \sum_{i=1}^{n} \lambda_i I_i^*(x_m) - y^*(x_m) \right| = \left| \sum_{i=1}^{n} \lambda_i I_i^*(x_m) \right| = 1 \Rightarrow \left\| \sum_{i=1}^{n} \lambda_i I_i^* - y^* \right\|.$$ 

so letting $m \to \infty$ and using Lemma 3.3.1, we get

$$\left\| \sum_{i=1}^{n} \lambda_i I_i^* + Y \right\| = \text{dist} \left( \sum_{i=1}^{n} \lambda_i I_i^*, Y \right) = \lim_{m \to \infty} \text{dist} \left( \sum_{i=1}^{n} \lambda_i I_i^*, Y_m \right) \geq 1,$$

thus $U$ is an isometry. We extend $U$ by density on $JT^*/Y$ and we still denote it as $U$ for convenience. It remains to show it is surjective. Indeed, let $f = \sum_{n=1}^{\infty} \lambda_n \sigma_n = \lim_{k \to \infty} U \left( \sum_{n=1}^{k} \lambda_n \sigma_n + Y \right)$. Then $f \in U[JT^*/Y]$. Since $JT^*/Y$ is a Banach space, then $U[JT^*/Y]$ is a Banach space, since they are isometric. Hence $U[JT^*/Y]$ is a closed subspace of $\ell_2(2^\mathbb{N})$, so $f \in U[JT^*/Y]$. We conclude that $U$ is surjective.

We immediately see a fundamental difference compared to $J$.

**Corollary 3.3.1.** $JT$ si not quasi-reflexive i.e. $JT^{**}/JT$ is infinite dimensional.
Proof. Since the basis of $\mathcal{J}T$ is boundedly complete, Proposition 1.5.7 yields
$$\dim(\mathcal{J}T^{**}/\mathcal{J}T) = \dim((\mathcal{J}T/Y)^*) = \dim(\ell_2(2^N)) = \dim(\ell_2(2^N)) = +\infty.$$  

\[ \square \]

Theorem 3.3.3. The following description for the second conjugate holds:
$$\mathcal{J}T^{**} = \mathcal{J}T \oplus [\sigma^{**} : \sigma \in 2^N].$$

In particular, any $x^{**} \in \mathcal{J}T^{**}$ can be uniquely written as
$$x^{**} = \hat{x} + \sum_{n=1}^{\infty} \lambda_n \sigma_n^{**},$$

where $x \in \mathcal{J}T$, $\{\lambda_n\}_{n \in \mathbb{N}} \in \ell_2$ and $\{\sigma_n\}_{n \in \mathbb{N}} \subseteq 2^N$.

Proof. By Proposition 3.1.6, we obtain $\hat{\mathcal{J}T} \cap [\sigma^{**} : \sigma \in 2^N] = \{0\}$. Since the basis of $\mathcal{J}T$ is boundedly complete, Proposition 1.5.7 yields it is enough to show that each $y^* \in Y^\perp$ can be written as
$$y^* = \sum_{n=1}^{\infty} \lambda_n \sigma_n^{**},$$

where $\{\lambda_n\}_{n \in \mathbb{N}} \in \ell_2$ and $\{\sigma_n\}_{n \in \mathbb{N}} \subseteq 2^N$. Indeed consider the surjective isometries $T$ and $U$ defined in Propositions 1.5.6 and 3.3.1 respectively and the surjective isometry $S$ identifying $\ell_2(2^N)$ with its conjugate. Let us define $L = TU^* : \ell_2(2^N) \to Y^\perp$. Then $L$ is surjective isometry as well. Since $L$ is surjective and bounded, it suffices to show $L(\mathcal{X}(\sigma)) = \sigma^{**}$, for any $\sigma \in 2^N$. Consider $\sigma \in 2^N$ and $I$ an infinite interval of $2^\mathbb{N}$. Then
$$L(\mathcal{X}(\sigma))(I^*) = TU^*S(\mathcal{X}(\sigma))(I^*) = S(\mathcal{X}(\sigma))(U(I^* + Y)) = U(I^* + Y)(\sigma) = \mathcal{X}(\sigma)(I)(\sigma) = \sigma^{**}(I^*) = L(\mathcal{X}(\sigma)) = \sigma^{**}.$$ 

The result follows immediately, since $\mathcal{J}T^* = [I^* : I$ infinite interval$]$.  

Let $X$ be a Banach space such that $X^*$ is non-separable. Goldstine’s Theorem yields that $X$ is $w^*$-dense in $X^{**}$. However, this is not enough to conclude that $\hat{X}$ is $w^*$-sequentially dense in $X^{**}$ since $X^*$ is not separable, hence the $w^*$-topology of $X^{**}$ is not metrizable. It has been shown however by H. Rosenthal and E. Odell, that if $X$ does not contain an isomorphic copy of $\ell_1$, this property holds. We will prove this property holds for $\mathcal{J}T$, without using the results of Rosenthal-Odell.
Theorem 3.3.4. \( J^T \) is \( w^* \)-sequentially dense in \( J^{T**} \) i.e. for any \( x^{**} \in J^{T**} \), there is sequence \( \{x_n\}_{n \in \mathbb{N}} \subseteq J^T \) such that \( x^{**} = \lim_{n \to \infty} x_n \).

Proof. Let \( x^{**} = \hat{x} + \sum_{j=1}^{\infty} \lambda_j \sigma_j^{**} \) and let us define \( x_1 = x \). For \( n > 1 \) we consider \( k_n \) to be the separation level of \( \sigma_1, ..., \sigma_n \). We define \( x_n = x + \sum_{j=1}^{n} \lambda_j e_{\sigma_j|_{k_n}} \). The sequence \( \{x_n\}_{n \in \mathbb{N}} \) is clearly bounded, since \( \{\lambda_n\}_{n \in \mathbb{N}} \in \ell_2 \). Hence, Proposition 1.1.5 implies it suffices to show \( I^*(x_n) \xrightarrow{n \to \infty} \sum_{j=1}^{\infty} \lambda_j \sigma_j^{**}(I^*) \) for any infinite interval \( I \). Indeed, consider an infinite interval \( I \). If there is \( i \in \mathbb{N} \) such that \( \sigma(I) = \sigma_i \), then for any \( n \geq n_0 \) we have that \( I^*(x_n) = \lambda_i = \sum_{n=1}^{\infty} \lambda_n \sigma_n^{**}(I^*) \).

Otherwise, we have \( I^*(x_n) = 0 = \sum_{j=1}^{\infty} \lambda_j \sigma_j^{**}(I^*) \) \( \forall n \in \mathbb{N} \). Therefore for any infinite interval \( I \) we have that \( I^*(x_n) \xrightarrow{n \to \infty} \sum_{j=1}^{\infty} \lambda_j \sigma_j^{**}(I^*) \) and the result follows.

3.4 \( J^T \) is \( \ell_2 \)-saturated

In this final section, we show \( J^T \) is \( \ell_2 \)-saturated i.e. all its infinite dimensional subspaces contain \( \ell_2 \). Our approach is based on [1].

At first we present a combinatorial result which turns out to be crucial for this approach. This kind of combinatorial results are frequently used in Banach Space Theory. Recall if \( M \) is a countable set, we denote \( [M] \) the set of all infinite subsets of \( M \). We will also denote

\[
M^{(2)} = \{(n, m) : n, m \in M, n < m\}
\]

Theorem 3.4.1. (Ramsey’s Theorem) Let \( A_1, ..., A_k \subseteq \mathbb{N} \) with \( \mathbb{N}^{(2)} = \bigcup_{i=1}^{k} A_i \). Then there is \( M \in [\mathbb{N}] \) and \( i \in \{1, ..., k\} \) such that \( M^{(2)} \subseteq A_i \).
Proof. Let $M_1 = \mathbb{N}$ and $m_1 \in M_1$. Then there is $M_2 \in [M_1]$ and $i_1 \in \{1, \ldots, k\}$ such that $\{m_1\}_{M_2}^{(2)} \subseteq A_{i_1}$. Consider $m_2 \in M_2$ with $m_2 > m_1$. Then there is $M_3 \in [M_2]$ and $i_2 \in \{1, \ldots, k\}$ with $\{m_2\}_{M_3}^{(2)} \subseteq A_{i_2}$. By induction, we find strictly increasing sequence $\{m_n\}_{n \in \mathbb{N}} \subseteq \mathbb{N}$, a decreasing sequence $\{M_n\}_{n \in \mathbb{N}} \subseteq [\mathbb{N}]$ and a sequence $\{t_n\}_{n \in \mathbb{N}} \subseteq \{1, \ldots, k\}$ with $m_p \in M_n \ \forall p \geq n$ and $\{m_p\}_{M_{n+1}}^{(2)} \subseteq A_{i_n}$, for any $n \in \mathbb{N}$. It is immediate that $\mathbb{N} = \bigcup_{i=1}^{k} \{n \in \mathbb{N} : \{m_n\}_{M_{n+1}}^{(2)} \subseteq A_i\}$. Therefore, there is $i \in \{1, \ldots, k\}$ such that the set $L = \{n \in \mathbb{N} : \{m_n\}_{M_{n+1}}^{(2)} \subseteq A_i\}$ is infinite. Let us define the set $M = \{m_n : n \in L\} \in [\mathbb{N}]$. Then, for $m_n, m_p \in M$ with $m_n < m_p$, we have that $n < p$. So $m_p \in M_{n+1}$ and $(m_n, m_p) \in A_i$. Therefore $M^{(2)} \subseteq A_i$ and the result is proved. \hfill \Box

We use Ramsey’s Theorem to prove an important Lemma.

**Lemma 3.4.1.** Let $\{x_n\}_{n \in \mathbb{N}}$ a block sequence in $\mathcal{JT}$ such that $\lim_{n \to \infty} I^*(x_n) = 0$, for any infinite interval $I$. Then for any $\epsilon > 0$, there is subsequence $\{x_n\}_{n \in M}$ such that for any bounded interval $S$, with $in(S) = \emptyset$, we have $|S^*(x_n)| \leq \epsilon$ for any $n \in M$, except at most one $n(S) \in M$, which depends on the choice of the interval $S$.

**Proof.** Let $\epsilon > 0$. We denote $K = \sup_{n \in \mathbb{N}} \{||x_n||\}$. For any $n \in \mathbb{N}$, we denote $\alpha_n = \min \{\text{supp}(x_n)\}$. Let us define the set

$$Q_n = \{t \in 2^{<\mathbb{N}} : |t| = \alpha_n \ \text{and} \ |S^*_t(x_n)| > \epsilon, \ \text{for some finite interval } S_t \}$$

with $in(S) = t$.

Then for any $n \in \mathbb{N}$, we have that

$$\epsilon^2 |Q_n| < \sum_{t \in Q_n} |S^*_t(x_n)|^2 \leq ||x_n||^2 \leq K^2 \Rightarrow |Q_n| \leq \frac{K^2}{\epsilon^2} \ \forall n \in \mathbb{N}$$

Hence, there is $\alpha \in \mathbb{N} \cup \{0\}$ and $L \in [\mathbb{N}]$ such that $|Q_n| = \alpha \ \forall n \in L$. If $\alpha = 0$ we trivially get the result for $(x_n)_{n \in L}$, since $Q_n = \emptyset \ \forall n \in L$. So, without loss of generality, we assume $\alpha \geq 1$. For $n \in L$, let us write $Q_n = \{t_{i,n} : 1 \leq i \leq \alpha\}$. Moreover, for $i, j \in \{1, \ldots, \alpha\}$, let us define

$$A_{i,j} = \{n, m \in L : n < m \ \text{and there is interval } S \ \text{with endpoints} \ t_{i,n} \ \text{and} \ t_{j,m}, \ \text{such that} \ |S^*(x_n)| > \epsilon\}$$

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Let \( A = \mathbb{N}^{(2)} \setminus \bigcup_{1 \leq i,j \leq \alpha} A_{i,j} \). Clearly \( \mathbb{N}^{(2)} = A \cup \bigcup_{1 \leq i,j \leq \alpha} A_{i,j} \). Thus, Ramsey’s Theorem implies that there is a \( M \in [\mathbb{N}] \) such that \( M^{(2)} \subseteq A \) or \( M^{(2)} \subseteq A_{i,j} \) for some \( i,j \in \{1,\ldots,\alpha\} \). Arguing by contradiction, let us assume there are \( i,j \in \{1,\ldots,\alpha\} \) with \( M^{(2)} \subseteq A_{i,j} \). Then for any \( n,k \in M \) with \( n+1 < k \), we have that \((n,k),(n+1,k) \in M^{(2)}\). We show there is interval with endpoints \( t_{i,n} \) and \( t_{i,n+1} \). Indeed, there is interval \( S_1 \) initiating at \( t_{i,n} \) and interval \( S_2 \) initiating at \( t_{i,n+1} \) which both end at \( t_{j,k} \), such that \( |S_1^*(x_n)| > \epsilon \) and \( |S_2^*(x_{n+1})| > \epsilon \). Therefore, the definition of the intervals implies that \( t_{i,n+1} \in S_1 \). Let us define the interval \( S_{n,n+1} \) initiating at \( t_{i,n} \) and ending at \( t_{i,n+1} \). Then \( |S_{n,n+1}^*(x_n)| = |S_1^*(x_n)| > \epsilon \). We define \( I = \bigcup_{n=1}^{\infty} S_{n,n+1} \). Clearly \( I \) is infinite interval and \( |I^*(x_n)| = |S_{n,n+1}^*(x_n)| > \epsilon \quad \forall n \in M \), which is a contradiction, since \( \lim_{n \to \infty} I^*(x_n) = 0 \). Hence \( M^{(2)} \subseteq A \). We finally show that the sequence \((x_n)_{n \in M}\) has the required property. Indeed, consider an interval \( S \) with \( in(S) = \emptyset \) and \( n,m \in M \) with \( n < m \) such that \( |S^*(x_n)| > \epsilon \) and \( |S^*(x_m)| > \epsilon \). Then, considering \( t_1 \in S \) with \( |t_1| = \alpha_n \) and \( t_2 \in S \) with \( |t_2| = \alpha_m \), the definition of \( Q_n \) and \( Q_m \) yields there are \( i,j \in \{1,\ldots,\alpha\} \) such that \( t_1 = t_{i,n} \) and \( t_2 = t_{j,m} \). So \((n,m) \in A_{i,j} \), which contradicts the fact that \( A \cap \bigcup_{1 \leq i,j \leq \alpha} A_{i,j} = \emptyset \). \( \square \)

Let us introduce some terminology. Any block sequence \((x_n)_{n \in \mathbb{N}} \subseteq \mathcal{JT}\), such that for given \( \epsilon > 0 \), there is some \( M \in [\mathbb{N}] \) such that for any bounded interval \( S \) with \( in(S) = \emptyset \), we have \( |S^*(x_n)| \leq \epsilon \quad \forall n \in M \) except at most one node \( n(S) \), will be called \( \epsilon \)-separated.

**Lemma 3.4.2.** Let \( \{y_n\}_{n \in \mathbb{N}} \subseteq \mathcal{JT} \) be a unitary block with \( \lim_{n \to \infty} I^*(x_n) = 0 \) for any infinite interval \( I \). Then for any \( \epsilon > 0 \), there is a subsequence \( \{y_n'\}_{n \in \mathbb{N}} \) such that for any \( n \in \mathbb{N} \) and \( \lambda_1,\ldots,\lambda_n \in \mathbb{R} \), there holds

\[
(\sum_{i=1}^{n} \lambda_i^2)^{1/2} \leq ||\sum_{i=1}^{n} \lambda_i y_i'|| \leq 2(1+\epsilon)(\sum_{i=1}^{n} \lambda_i^2)^{1/2}
\]

**Proof.** Let \( \epsilon > 0 \). Consider a null sequence \( \{\epsilon_k\}_{k \in \mathbb{N}} \subseteq \mathbb{R} \). By Lemma 3.4.1, we may consider a decreasing sequence \( \{M_k\}_{k \in \mathbb{N}} \subseteq [\mathbb{N}] \) such that the sequence \( \{y_n\}_{n \in M_k} \) is \( \epsilon_k \)-separated for any \( k \in \mathbb{N} \). Hence, for any \( k \in \mathbb{N} \) and any bounded interval \( S \) with \( in(S) = \emptyset \), we have that \( |S(y_n)| \leq \epsilon_k \quad \forall n \in M_k \).
except at most one node \( n(S) \in M_k \). For any \( n \in \mathbb{N} \), we define \( \alpha_n = \min \{ \text{supp}(x_n) \} \) and \( m_n = 2^{\alpha_n} \). We find strictly increasing sequences of positive integers \( \{p_n\}_{n \in \mathbb{N}} \) and \( \{k_n\}_{n \in \mathbb{N}} \) such that

\[
p_n \in M_{k_n} \quad \forall n \in \mathbb{N} \quad \text{and} \quad \sum_{n=1}^{\infty} m_n \left( \sum_{l=n+1}^{\infty} \epsilon_{k_l}^2 \right) < \epsilon^2
\]

We define \( k_1 = 1 \) and consider \( p_1 \in M_{k_1} \). Since \( \epsilon_n \xrightarrow{n \to \infty} 0 \), there is \( L_1 \in [\mathbb{N}] \) such that

\[
\sum_{r \in L_1} \epsilon_r^2 < \frac{\epsilon^2}{2m_1}
\]

We consider \( k_2 \in L_1 \) with \( k_2 \in L_1 \) and \( p_2 \in M_{k_2} \) with \( p_2 > p_1 \). Since \( \epsilon_n \xrightarrow{n \in L_1} 0 \) there is \( L_2 \in [L_1] \) such that

\[
\sum_{r \in L_2} \epsilon_r^2 < \frac{\epsilon^2}{2^2m_2}
\]

We consider \( k_3 \in L_2 \) with \( k_3 > k_2 \) and \( p_3 \in M_{k_3} \) with \( p_3 > p_2 \). Continuing inductively we construct strictly increasing sequences of positive integers \( \{k_n\}_{n \in \mathbb{N}} \) and \( \{p_n\}_{n \in \mathbb{N}} \), such that \( p_n \in M_{k_n} \quad \forall n \in \mathbb{N} \), and a decreasing sequence \( \{L_n\}_{n \in \mathbb{N}} \subseteq [\mathbb{N}] \) with \( k_n \in L_n \) and \( \sum_{r \in L_n} \epsilon_r^2 < \frac{\epsilon^2}{2^n m_n} \quad \forall n \in \mathbb{N} \). Then \( k_l \in L_n \quad \forall l \geq n + 1 \), so

\[
\sum_{l=n+1}^{\infty} \epsilon_{k_l}^2 \leq \sum_{r \in L_n} \epsilon_r^2 < \frac{\epsilon^2}{2^n m_n} \quad \forall n \in \mathbb{N} \quad \Rightarrow \quad \sum_{n=1}^{\infty} m_n \left( \sum_{l=n+1}^{\infty} \epsilon_{k_l}^2 \right) < \epsilon^2
\]

Therefore, the sequences \( \{k_n\}_{n \in \mathbb{N}} \) and \( \{p_n\}_{n \in \mathbb{N}} \) have the required properties.

For any \( n \in \mathbb{N} \), we define \( y'_n = y_{p_n}, b_n = \alpha_{p_n}, \delta_n = \epsilon_{k_n} \). Then the sequence \( \{y'_n\}_{n \geq k} \) is \( \delta_k \)-separated for any \( k \in \mathbb{N} \). Indeed, for any \( k \in \mathbb{N} \), we have \( p_n \in M_k \quad \forall n \geq k \). Thus, the sequence \( \{y'_n\}_{n \geq k} \) is subsequence of \( \{y_n\}_{n \in M_k} \) and so it is \( \delta_k \)-separated. Moreover, we have that

\[
\sum_{n=1}^{\infty} m_n \left( \sum_{l=n+1}^{\infty} \delta_l^2 \right) < \epsilon^2
\]

For any bounded interval \( S \), we define \( i(S) = \min \{ n \in \mathbb{N} : |in(S)| \leq b_n \} \).

The interval \( S \) will be called regular if \( |in(S)| = b_i(S) \). Consider a regular
interval $S$. We define the set $A(S) = \{ n \in \mathbb{N} : |S(y'_n)| > \delta_n \}$. If $A(S) \neq \emptyset$, we define $\lambda(S) = \min A(S)$. Otherwise, we define $\lambda(S) = +\infty$. We show that

$$\sum_{n \notin \lambda(S)} |S^*(y'_n)|^2 \leq \sum_{n \geq i(S)} \delta_n^2$$

Assume $A(S) = \{ n_1 < n_2 < \ldots < n_j < n_{j+1} < \ldots \}$. It is clear that $\lambda(S) = n_1$ and $i(S) \leq n_1$. Since for any $j \in \mathbb{N}$ the sequence $(y'_n)_{n \geq n_j}$ is $\delta_{n_j}$-separated and $|S^*(y'_{n_j})| > \delta_{n_j}$, we have that $|S^*(y'_{n_{j+1}})| \leq \delta_{n_j}$. So we obtain

$$\sum_{n \notin \lambda(S)} |S^*(y'_n)|^2 = \sum_{n \geq i(S); n \notin \lambda(S)} |S^*(y'_n)|^2$$

$$= \sum_{j=2}^{\infty} |S^*(y'_{n_j})|^2 + \sum_{n \geq i(S); n \notin A(S)} |S^*(y'_n)|^2$$

$$\leq \sum_{j=1}^{\infty} \delta_{n_j}^2 + \sum_{n \geq i(S); n \notin A(S)} \delta_n^2 = \sum_{n \geq i(S)} \delta_n^2.$$

Consider now $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\sum_{i=1}^{n} \lambda_i^2 = 1$. With a similar argument to Remark 2.4.1 we can see that $1 \leq ||\sum_{i=1}^{n} \lambda_i y'_i||$. For the opposite direction, let $U$ a finite family of pairwise disjoint intervals. For $S \in U$, we have that $S = S_0 \cup S'$, where

$$S_0 = \{ t \in S : |t| < b_i(S) \}$$
$$S' = \{ t \in S : b_i(S) \leq |t| \}$$

Clearly, one of these sets can be empty. Without loss of generality, we assume none of them, and the sets we define below is empty. Hence, $S_0, S'$ are intervals and in particular $S'$ is regular. We have $S' = S_1 \cup S''$ where

$$S_1 = \{ t \in S : b_i(S) \leq |t| < b_i(S)+1 \}$$
$$S'' = \{ t \in S : b_i(S)+1 \leq |t| \}$$
The interval $S''$ is regular and $S'' = S_1'' \cup S_2'' \cup S_3''$, where

$$
S_1'' = \{ t \in S : b_{i(S)+1} \leq |t| < b_{\lambda(S'')} \}
$$

$$
S_2'' = \{ t \in S : b_{\lambda(S'')} \leq |t| < b_{\lambda(S'') + 1} \}
$$

$$
S_3'' = \{ t \in S : b_{\lambda(S'')} + 1 \leq |t| \}
$$

Let us note, in the case of $\lambda(S'') = +\infty$, we will denote $b_{\lambda(S'')} = +\infty$. We conclude $S = S_0 \cup S_1 \cup S_1'' \cup S_2'' \cup S_3''$. Therefore, defining $x = \sum_{i=1}^{n} \lambda_i y'_i$, we get

$$
\sum_{S \in \mathcal{U}} |S^*(x)|^2 = \sum_{S \in \mathcal{U}} |S_0^*(x) + S_1^*(x) + S_1''^*(x) + S_2''^*(x) + S_3''^*(x)|^2
$$

$$
\leq 4 \sum_{S \in \mathcal{U}} (|S_0^*(x)|^2 + |S_1^*(x)|^2 + |S_2''^*(x)|^2) + 4 \sum_{S \in \mathcal{U}} |S_1''^*(x) + S_3''^*(x)|^2
$$

Let us denote $R$ for any of the sets $S_0, S_1, S_2''$. We notice that $R(x) = \lambda_i R(y'_i)$, for at most one $i \in \{1, \ldots, n\}$. Hence $|R(x)|^2 = \sum_{i=1}^{n} \lambda_i^2 |R(y'_i)|^2$ and

$$
\sum_{S \in \mathcal{U}} (|S_0^*(x)|^2 + |S_1^*(x)|^2 + |S_2''^*(x)|^2) =
$$

$$
= \sum_{S \in \mathcal{U}} \sum_{i=1}^{n} (\lambda_i^2 (|S_0^*(y'_i)|^2 + |S_1^*(y'_i)|^2 + |S_2''^*(y'_i)|^2))
$$

$$
= \sum_{i=1}^{n} \lambda_i^2 (\sum_{S \in \mathcal{U}} (|S_0^*(y'_i)|^2 + |S_1^*(y'_i)|^2 + |S_2''^*(y'_i)|^2))
$$

$$
\leq \sum_{i=1}^{n} \lambda_i^2 |y'_i|| = \sum_{i=1}^{n} \lambda_i^2 = 1
$$

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Moreover, for any $S \in \mathcal{U}$, we have that
\[
|S''_1(x) + S''_3(x)|^2 = | \sum_{i=1, i \neq \lambda(S'')} \lambda_i S''(y_i')|^2 \\
\leq ( \sum_{i=1, i \neq \lambda(S'')} \lambda_i^2)( \sum_{i=1, i \neq \lambda(S'')} |S''(y_i')|^2) \\
\leq \sum_{n \neq \lambda(S'')} |S''(y_n')|^2 \leq \sum_n \delta_n^2
\]

For $k \in \mathbb{N}$, let us define $\mathcal{U}_k = \{ S \in \mathcal{U} : i(S) = k \}$. Then for $S \in \mathcal{U}_k$, we have that $i(S'') = k + 1$ so
\[
\sum_{S \in \mathcal{U}} |S''_1(x) + S''_3(x)|^2 = \sum_{k=1}^\infty \sum_{S \in \mathcal{U}_k} |S''_1(x) + S''_3(x)| \\
= \sum_{k=1}^\infty \sum_{S \in \mathcal{U}_k} \sum_{n \geq i(S'')} \delta_n^2 = \sum_{k=1}^\infty \sum_{S \in \mathcal{U}_k} \sum_{n \geq k+1} \delta_n^2 \leq \sum_{k=1}^\infty m_k (\sum_{n=k+1}^\infty \delta_n^2) < \epsilon^2
\]

Combining these estimates, for any family $\mathcal{U}$ of bounded and pairwise disjoint intervals, we get
\[
\sum_{S \in \mathcal{U}} |S''(x)|^2 < 4 + 4\epsilon^2 < 4(1 + \epsilon)^2 \Rightarrow ||\sum_{i=1}^n \lambda_i y_i'|| \leq 2(1 + \epsilon)
\]

Hence
\[
1 \leq ||\sum_{i=1}^n \lambda_i y_i'|| \leq 2(1 + \epsilon)
\]

and the result follows.

We finally prove the main result of this section.

**Theorem 3.4.2.** $\mathcal{T}$ is $\ell_2$-saturated. In particular, for any infinite dimensional subspace $Y$, there is sequence $\{x_n\}_{n \in \mathbb{N}}$ in $Y$, such that for any $\epsilon > 0$ there is one subsequence $\{x''_n\}_{n \in \mathbb{N}}$ such that for any $n \in \mathbb{N}$ and $\lambda_1, ..., \lambda_n \in \mathbb{R}$, there holds
\[
(1 - \epsilon)\left(\sum_{i=1}^n \lambda_i^2\right)^{1/2} \leq ||\sum_{i=1}^n \lambda_i x''_i|| \leq (2 + 3\epsilon)\left(\sum_{i=1}^n \lambda_i^2\right)^{1/2}
\]
Proof. Let $Y$ be infinite dimensional subspace of $\mathcal{J} \mathcal{T}$. Since $\ell_1$ does not embed isomorphically in $\mathcal{J} \mathcal{T}$, we may use Proposition 1.3.3, to find a weakly null and unitary sequence $\{x_n\}_{n \in \mathbb{N}}$ in $Y$. Let $\epsilon > 0$, then, using a sliding hump argument, we may find a subsequence $\{x'_n\}_{n \in \mathbb{N}}$ and a unitary block $(y_n)_{n \in \mathbb{N}}$ such that

$$\sum_{n=1}^{\infty} ||x'_n - y_n||^2 < \epsilon^2$$

Clearly $I^*(x'_n) \xrightarrow{n \to \infty} 0$ and by (3.4), we obtain $||x'_n - y_n|| \xrightarrow{n \to \infty} 0$. So, for any infinite interval $I$ we have

$$|I^*(y_n)| \leq |I^*(x_n)| + ||I^*|||x'_n - y_n|| \quad \forall n \in \mathbb{N} \Rightarrow$$

$$\lim_{n \to \infty} I^*(y_n) = 0$$

Therefore, Lemma 3.4.2 implies that for any $\epsilon > 0$ there is a subsequence $\{y'_n\}_{n \in \mathbb{N}}$ such that for any $n \in \mathbb{N}$ and $\lambda_1, ..., \lambda_n \in \mathbb{R}$, there holds

$$\left(\sum_{i=1}^{n} \lambda_i^2\right)^{1/2} \leq \left|\sum_{i=1}^{n} \lambda_i y'_i\right| \leq 2(1 + \epsilon)(\sum_{i=1}^{n} \lambda_i^2)^{1/2}$$

Let us denote $\{x''_n\}_{n \in \mathbb{N}}$ the corresponding subsequence of $\{x'_n\}_{n \in \mathbb{N}}$. Then for any $n \in \mathbb{N}$ and $\lambda_1, ..., \lambda_n \in \mathbb{R}$, we have

$$\left|\sum_{i=1}^{n} \lambda_i x''_i\right| \leq \sum_{i=1}^{n} |\lambda_i||x''_i - y'_i| + \left|\sum_{i=1}^{n} \lambda_i y'_i\right|$$

$$\leq \left(\sum_{i=1}^{n} \lambda_i^2\right)^{1/2}\left(\sum_{i=1}^{n} ||x''_i - y'_i||^2\right)^{1/2} + 2(1 + \epsilon)(\sum_{i=1}^{n} \lambda_i^2)^{1/2}$$

$$\leq (2 + 3\epsilon)(\sum_{i=1}^{n} \lambda_i^2)^{1/2}$$
and

$$\| \sum_{i=1}^{n} \lambda_i y_i' \| - \sum_{i=1}^{n} |\lambda_i| |x_i'' - y_i'| \leq \| \sum_{i=1}^{n} \lambda_i x_i'' \| \Rightarrow$$

$$\Rightarrow \left( \sum_{i=1}^{n} \lambda_i^2 \right)^{1/2} - \left( \sum_{i=1}^{n} \lambda_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} (\|x_i'' - y_i'\|^2)^{1/2} \right) \leq \| \sum_{i=1}^{n} \lambda_i x_i'' \| \Rightarrow$$

$$\Rightarrow (1 - \epsilon) \left( \sum_{i=1}^{n} \lambda_i^2 \right)^{1/2} \leq \| \sum_{i=1}^{n} \lambda_i x_i'' \|$$

Putting all the pieces together, we obtain

$$(1 - \epsilon) \left( \sum_{i=1}^{n} \lambda_i^2 \right)^{1/2} \leq \| \sum_{i=1}^{n} \lambda_i x_i'' \| \leq (2 + 3\epsilon) \left( \sum_{i=1}^{n} \lambda_i^2 \right)^{1/2}$$

and the result comes from Remark 1.3.1. \(\square\)
Bibliography


