M210T - Emerging Scholars Seminar
Worksheet 11
March 8, 2010

1. Find the arc length for the following curves for $0 \leq t \leq 2$.

$$\text{a)} \quad \begin{array}{l} x = \frac{t}{1+1} \\ y = \ln(1+t) \end{array} \qquad \text{b)} \quad \begin{array}{l} x = e^t \cos(\pi t/2) \\ y = e^t \sin(\pi t/2) \end{array} \qquad \text{c)} \quad \begin{array}{l} x = \sin^2(\pi t) \\ y = \cos^2(\pi t) \end{array}$$

2. Find the area under the following curves for $0 \leq t \leq \pi$.

$$\text{a)} \quad \begin{array}{l} x = 5\cos t \\ y = 3\sin t \end{array} \qquad \text{b)} \quad \begin{array}{l} x = \cos^3 t \\ y = \sin^3 t \end{array} \qquad \begin{array}{l} x = t^2 \\ y = t^3 \end{array}$$

3. Sketch the graphs for the following polar equations.

$$r = -2\cos\theta, \quad r = 1 + 2\cos\theta, \quad r = 2\cos(3\theta), \quad r^2 - 5r + 6 = 0, \quad r = 2^\theta$$

4. Find a formula to calculate the area of the region between (r, θ) and the origin (ask me to draw a picture), for $\alpha \leq \theta \leq \beta$.

5. The Isoparemetric Problem:
Consider all smooth closed curves in the plane having a given length. Is there one which encloses a largest area?
Assumptions: The curve has no crossings, and the curve is convex (Why can you make these assumptions?).
1) Show that if A is the area enclosed by the curve, then

$$A = \int_0^\pi y\left(\frac{dx}{ds}\right) ds + \int_\pi^{2\pi} y\left(\frac{dx}{ds}\right) ds.$$

Let $A_1 = \int_0^\pi y\left(\frac{dx}{ds}\right) ds$ (we will show that $A_1 \leq \pi/2$).

2) Show that for any real numbers a and b, $ab \leq \frac{a^2+b^2}{2}$, and use this to show that

$$A_1 \leq \frac{1}{2}\int_0^\pi \left(y^2 + \left(\frac{dx}{ds}\right)^2\right) ds.$$

3) Verify that $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$ using the fact that s is arc length, hence:

$$A_1 \leq \frac{1}{2}\int_0^\pi \left[y^2 + 1 - \left(\frac{dy}{ds}\right)^2\right] ds.$$

4) Write $y(s) = u(s) \cdot \sin(s)$, $0 \le s \le \pi$. Find $\frac{dy}{ds}$ and show that

$$A_1 \le \frac{1}{2} \int_0^{\pi} \left[u^2 \cdot (\sin^2(s) - \cos^2(s)) - 2u\frac{du}{ds} \cdot \sin(s)\cos(s) - \left(\frac{du}{ds}\right)^2 \cdot \sin^2(s) + 1 \right] ds.$$

5) Use the fact that $\frac{d}{ds}(u^2) = 2u\frac{du}{ds}$ and integration by parts to show that

$$A_1 \le \frac{1}{2} \in_0^{\pi} \left[1 - \left(\frac{du}{ds}\right)^2 \cdot \sin^2(s) \right] ds \le \frac{\pi}{2}.$$

When does equality hold?

In (2), $ab = \frac{a^2+b^2}{2}$ if and only if $a = b$, hence

$$A_1 = \frac{1}{2} \int_0^{\pi} \left[1 - \left(\frac{du}{ds}\right)^2 \cdot \sin^2(s) \right] ds = \frac{\pi}{2}$$

only when $\frac{du}{ds} = 0$; i.e. $u = C$ for some constant C. Thus, in (4), this yields $y(s) = C \cdot \sin(s)$, and $y = \frac{dx}{ds} = \sqrt{1 - \left(\frac{dy}{ds}\right)^2}$. So $y = \pm\sin(s)$, and thus $x(s) = \mp\cos(s) + C$.

6. * Due to a mistake in the manufacturing process, an analog clock was made so that the hour hand and minute hand that both point to twelve at midnight, but turn at the wrong speeds. The hour hand turns at the speed that the minute hand is supposed to turn, and the minute hand turns at the speed that the hour hand is supposed to turn. At how many times during a twelve-hour interval does the clock display a time that makes sense (in terms of the placement of the hour hand and minute hand), but is not the correct time?