M210T - Emerging Scholars Seminar
Worksheet 12
March 24, 2010

1. Some vector spaces (such as $\mathbb{R}^3$) are also *inner product spaces.* This means that there is an inner product $(\mathbf{a}, \mathbf{b})$ (this is a function of two vectors, not an ordered pair; you can think of it as $f(\mathbf{a}, \mathbf{b})$ if you want) that satisfies the following:
a) $(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{a})$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ (in general, this condition reads $(\mathbf{a}, \mathbf{b}) = \overline{(\mathbf{b}, \mathbf{a})}$, but we'll stick with real numbers).
b) $(k\mathbf{a}, \mathbf{b}) = k(\mathbf{a}, \mathbf{b})$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, $k \in \mathbb{R}$.
c) $(\mathbf{a} + \mathbf{b}, \mathbf{c}) = (\mathbf{a}, \mathbf{c}) + (\mathbf{b}, \mathbf{c})$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$.
d) $(\mathbf{a}, \mathbf{a}) \geq 0$ for all $\mathbf{a} \in \mathbb{R}^3$, and $(\mathbf{a}, \mathbf{a}) = 0$ if and only if $\mathbf{a} = 0$.

   You will learn Friday about the *dot product* for vectors in $\mathbb{R}^3$. For two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, the dot product is defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

   Show that the dot product is an inner product.

2. Inner products (the dot product in particular) allow us to talk about length of vectors (and therefore distance), angles, and orthogonality. Show that for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$,
a) $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$ (In a general inner product space, length in an inner product space is defined by $|\mathbf{v}| = (\mathbf{v}, \mathbf{v})^{1/2}$).
b) $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$, where θ is the angle between the two vectors (Hint: remember the Law of Cosines?) (In a general inner product space, we can define the angle between vectors $\mathbf{u}$ and $\mathbf{v}$ by $\theta = \cos^{-1}\left(\frac{(\mathbf{u}, \mathbf{v})}{|\mathbf{u}||\mathbf{v}|}\right)$).

   Determine a condition involving the dot product under which $\mathbf{a}$ and $\mathbf{b}$ are perpendicular (In a general inner product space, this condition, which involves the inner product, defines *orthogonal* vectors, which is the analogue to perpendicular vectors).

3. Using the last part of (2), find a vector that is perpendicular to both $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ (this involves solving a system of two equations in three unknowns, so there are infinitely many solutions. Be careful that you don't divide by any of the a_i and b_i since we don't know that they're nonzero).

4. * Define

$$S = \frac{1}{10^1} + \frac{1}{10^2} + \frac{2}{10^3} + \frac{3}{10^4} + \cdots + \frac{F_n}{10^n} + \cdots$$

   where the denominators of the fractions are the powers of ten, and the numerators are the terms of the Fibonacci sequence. What does S look like when written as a decimal? Is this decimal repeating or non-repeating? (You might want to look back at what we've done on the Fibonacci sequence so far).