M210T - Emerging Scholars Seminar
Worksheet 5
February 8, 2010

1. Do the following converge or diverge? Why? If it converges, can you determine what it converges to?

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}, \quad \sum_{n=0}^{\infty} \frac{4^n}{n!}, \quad \sum_{n=1}^{\infty} \sqrt[n]{n}, \quad \sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)}{n}, \quad \sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n, \quad \sum_{n=1}^{\infty} \frac{\ln n}{n^p}$$

2. Compare (I'm intentionally vague here) the following functions ($d, k > 0, c > 1$): $\ln(n), n^k, n^n, n!, d, c^n$. If $p(x)$ is a polynomial, where would $p(n)$ fit into this hierarchy? Keeping these comparisons in mind will make the Comparison Test more useful.

3. Does $\displaystyle\sum_{n=1}^{\infty}(\sqrt[n]{n} - 1)$ converge or diverge? (Hint: Consider the previous problem, particularly $p(x)$ and $\ln(x)$.)

4. Show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ converges. (Hint: The integral test is more useful than you think.)

5. *Cauchy's Condensation Test:* (this is a test you will not see in the book)
If $\{a_k\}_{k=1}^{\infty}$ is a decreasing positive sequence and $s_n = \sum_{k=0}^{n} a_k$, show that

$$\frac{1}{2}\left(a_1 + 2a_2 + 4a_4 + \ldots + 2^n a_n\right) \le s_{2^n} \le \left(a_1 + 2a_2 + 4a_4 + \ldots + 2^{n-1}a_{2^{n-1}}\right) + a_{2^n}.$$

It follows that $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges (Why?). This gives us another way to show (among other things) that $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $p > 1$.

6. * You have an infinite supply of ping-pong balls, numbered uniquely with positive integers (#1, #2, #3, ...). One minute before noon you add the first ten balls to a bag of infinite volume, and you remove ball #1. 30 seconds before noon you add the next ten balls, #11 through #20, and you remove ball #2. 15 seconds before noon, you add balls #21 through #30 and remove ball #3. You continue in this fashion adding the next ten balls and removing the lowest numbered ball at $60/2^n$ seconds before noon. How many ping-pong balls are in the bag at noon?