M210T - Emerging Scholars Seminar
Worksheet 7
February 22, 2010

1. Find the IOC (Interval of Convergence, not the International Olympic Committee) of the following power series:

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2}, \quad \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}, \quad \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^2 5^n}$$

   Find a power series whose IOC is a bounded open interval.

2. Find the ROC of the following power series:

$$\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}, \quad \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n, \quad \sum_{n=1}^{\infty} \frac{(2n)!}{n^n} x^n$$

$$\sum_{n=0}^{\infty} c_n (x-a)^n \text{ where } \begin{cases} \lim_{n\to\infty} \sqrt[n]{c_n} = \infty \text{ (Find the IOC also)} \\ \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L, \text{ and } 0 < L < \infty \\ \lim_{n\to\infty} \sqrt[n]{c_n} = 0 \text{ (Find the IOC also)} \end{cases}$$

3. Find a power series centered at $a = 0$ for each of the following functions:

$$\frac{1}{1-x}, \quad \frac{1}{2+x}, \quad \ln(1-x), \quad \frac{1}{(1-x)^2}, \quad \frac{1}{1-x^2}, \quad \frac{x^4}{(1-x)^2}, \quad \frac{1}{(1-x)^3}, \quad \tan^{-1} x$$

4. (This problem involves a lot of challenging algebra, but it's an interesting application of power series.)
   Recall that the Fibonacci sequence $\{F_n\}$ is defined by $F_0 = F_1 = 1$ and $F_n = F_{n-2} + F_{n-1}$ for $n \geq 2$. Consider the function $f(x) = \sum_{n=0}^{\infty} F_n x^n$ (you can find the ROC using a problem from an earlier worksheet, but that's not necessary for this exercise). The following steps lead us to a non-recursive formula for the Fibonacci numbers.
   a) Show that $f(x) = 1 + xf(x) + x^2 f(x)$.
   b) Use (a) to show that $f(x) = \frac{1}{(1-\alpha x)(1-\beta x)}$ for some real numbers α and β (and find α and β).
   c) Use (b) to write $f(x)$ as a power series centered at $x = 0$ with explicit coefficients (this takes some work). Compare this to the original definition of $f(x)$. We will see later that if $\sum_{n=k}^{\infty} b_n (x-a)^n = \sum_{n=k}^{\infty} c_n (x-a)^n$ on some interval, then $b_n = c_n$ for all $n \geq k$. Apply this theorem to derive a closed-form solution (Binet's formula) for the n-th Fibonacci number.

5. (This is a sketch of the proof of the theorem used in part (c) of the problem above.) Suppose that the function f has a power series representation at a with $R > 0$; i.e., on some interval,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

- Find c_0 in terms of f.
- Write a power series for $f'(x)$. Use this to find c_1.
- Write a power series for $f''(x)$. Use this to find c_2.
- Find a general formula for $f^{(n)}(x)$. Use this to find c_n.

What conclusion can you make regarding the claim made in part (c) of the previous problem?

6. * A spider stands on a corner of a square, and a fly sits on the opposite corner. The spider moves from corner to corner along the edges of the square, choosing her path (left or right) randomly. What is the expected (average) number of moves the spider will make before reaching the fly?