M210T - Emerging Scholars Seminar
Worksheet 9
March 1, 2010

1. (Did you do this one last time? If not, try it now.) Find the power series centered at $a = 0$ for the following functions:

$$\cos x, \quad e^x, \quad \int_0^x e^{t^2} dt, \quad x \sin x$$

2. Find the power series centered at $a = \pi$ for the following functions:

$$\cos x, \quad x^3, \quad e^x$$

3. Find a function which is defined for all real number but does not have a power series representation centered at $a = 0$. What criteria ensure that $f(x)$ has a power series representation on an interval centered at 0?

4. Write the Maclaurin series for $\sin x$ and $\cos x$. Using these, compute $i \sin(x) + \cos(x)$.

5. Write the Maclaurin series for e^x and compute e^{ix}.

6. How many terms of the MacLaurin series of $\sin x$ must you take to ensure that $\left|\cos x - \sum_{n=0}^{N} c_n x^n\right| <$ .001 for a particular x?

7. Taylor's Error Estimation Theorem (note that we do not explicitly use Taylor Series representations in this proof; i.e. we never assume that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$):
Let f be a function on $(-R, R)$ so that $f^{(k)}(x)$ exists for all $k = 1, 2, \ldots, n+1$ and $|f^{(n+1)}(t)| \leq M$ for all $t \in (-R, R)$.
a) Verify that

$$\frac{f^{(k)}(0)}{k!} x^k = \frac{1}{(k-1)!} \int_0^x f^{(k)}(t)(x-t)^{k-1} dt - \frac{1}{k!} \int_0^x f^{(k+1)}(t)(x-t)^k dt$$

by computing the second integral using integration by parts.

b) Define the polynomial

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n.$$

Write $T_n(x)$ using the formula from (a). Simplify the expression and solve for $f(x)$. You should end up with an expression like

$$f(x) = T_n(x) + R_n(x)$$

where $R_n(x) = \underline{\ ?\ }$.

c) Derive the following bound for $R_n(x)$:

$$|R_n(x)| \leq M \frac{|x|^{n+1}}{(n+1)!}.$$

d) What does this tell us?

e) What if we consider a power series centered at $a \neq 0$? Does this proof still work?

8. * Euler's Formula for convex polyhedrons states that $V - E + F = 2$ where V is the number of verticies, E is the number of edges, and F is the number of faces of the polyhedron (any idea how to prove this?). A Platonic solid is a polyhedron whose faces are congruent regular polygons, and which has the same number of edges meeting at each vertex. Prove that there are only finitely many Platonic solids. Can you list them?