①

$$I = \int \sin^6(3x)\,dx$$

$$= \int (\sin^2(3x))^3\,dx$$

$$= \int \left(\tfrac{1}{2}(1-\cos(6x))\right)^3 dx$$

$$= \frac{1}{8}\int \left(1 - 3\cos(6x) + 3\cos^2(6x) - \cos^3(6x)\right) dx$$

$$= \frac{1}{8}\left(x - 3\cdot\frac{1}{6}\sin(6x) + 3\int \frac{1}{2}(1+\cos(12x))\,dx - \int \cos^2(6x)\cos(6x)\,dx\right.$$

$$= \frac{1}{8}\left(x - \frac{1}{2}\sin(6x) + \frac{3}{2}\left(x + \frac{1}{12}\sin(12x)\right) - \int (1-\sin^2(6x))\cos(6x)\,dx\right)$$

$$= \frac{1}{8}\left(\frac{5}{2}x - \frac{1}{2}\sin(6x) + \frac{1}{8}\sin(12x) - \frac{1}{6}\int (1-u^2)\,du\right)$$

$$= \frac{1}{8}\left(\frac{5}{2}x - \frac{1}{2}\sin(6x) + \frac{1}{8}\sin(12x) - \frac{1}{6}\left(u - \frac{1}{3}u^3\right)\right) + C$$

$$= \frac{1}{8}\left(\frac{5}{2}x - \frac{1}{2}\sin(6x) + \frac{1}{8}\sin(12x) - \frac{1}{6}\sin(6x) + \frac{1}{18}\sin^3(6x)\right) + C$$

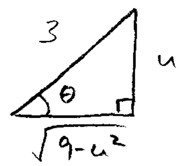
$$= \frac{5}{16}x - \frac{1}{12}\sin(6x) + \frac{1}{64}\sin(12x) + \frac{1}{144}\sin^3(6x) + C$$

② $\int \sqrt{9e^{6t} - e^{12t}}\, dt$

$= \int \sqrt{e^{6t}(9 - e^{6t})}\, dt$

$= \int e^{3t} \sqrt{9 - e^{6t}}\, dt$ $\qquad u = e^{3t} \qquad du = 3e^{3t}\, dt$

$= \frac{1}{3} \int \sqrt{9 - u^2}\, du$



$u = 3 \sin\theta \qquad \sqrt{9-u^2} = 3\cos\theta$

$du = 3\cos\theta\, d\theta$

$= \frac{1}{3} \int 3\cos\theta \cdot 3\cos\theta\, d\theta$

$= 3 \int \cos^2\theta\, d\theta$

$= 3 \int \frac{1}{2}(1 + \cos 2\theta)\, d\theta$

$= \frac{3}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$

$= \frac{3}{2}\left(\theta + \frac{1}{2} \cdot 2\sin\theta\cos\theta\right) + C$

$= \frac{3}{2}\left(\sin^{-1}\left(\frac{u}{3}\right) + \frac{u}{3} \cdot \frac{\sqrt{9-u^2}}{3}\right) + C$

$= \frac{3}{2}\left(\sin^{-1}\left(\frac{e^{3t}}{3}\right) + \frac{e^{3t}\sqrt{9-e^{6t}}}{9}\right) + C$

$= \frac{1}{6}\left(9\sin^{-1}\left(\frac{e^{3t}}{3}\right) + e^{3t}\sqrt{9-e^{6t}}\right) + C$

③ $I = \displaystyle\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\sin x \cos x}\,dx$

$= \displaystyle\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\frac{\sin x}{\cos x}\cdot \cos^2 x}\,dx$

$= \displaystyle\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\tan x}\cdot \sec^2 x\,dx$ $u = \tan x$ $du = \sec^2 x\,dx$

$= \displaystyle\int_{1}^{\sqrt{3}} \frac{\ln u}{u}\,du$ $\left(w = \ln u \quad dw = \frac{1}{u}\,du\right)$

$= \frac{1}{2}(\ln u)^2 \Big|_1^{\sqrt{3}} = \frac{1}{2}\left((\ln\sqrt{3})^2 - (\ln 1)^2\right) = \frac{1}{2}\left(\frac{1}{2}\ln 3\right)^2 = \boxed{\frac{1}{8}(\ln 3)^2}$

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④ $I = \displaystyle\int \frac{3x^2-2}{x^2-2x-8}\,dx$

$$\begin{array}{r} 3 \\ x^2-2x-8 \overline{\smash{)}\,3x^2+0x-2} \\ \underline{3x^2-6x-24} \\ 6x+22 \end{array}$$

$= \displaystyle\int 3 + \frac{6x+22}{(x-4)(x+2)}\,dx$

$= 3x + \displaystyle\int \left(\frac{A}{x-4} + \frac{B}{x+2}\right)dx$ where $A(x+2) + B(x-4) = 6x+22$

$Ax + Bx = 6x$ | $2A - 4B = 22$

$A + B = 6$ | $A - 2B = 11$

$A + B = 6$

$-A + 2B = -11$

$3B = -5$

$B = -\frac{5}{3}$

$A = 11 - 2\left(-\frac{5}{3}\right) = \frac{23}{3}$

$A = \frac{22}{3}$

$= 3x + \displaystyle\int \left(\frac{22}{3}\frac{1}{x-4} - \frac{5}{3}\frac{1}{x+2}\right)dx$

$= 3x + \frac{22}{3}\ln|x-4| - \frac{5}{3}\ln|x+2| + C$

⑤ $I = \int \frac{1}{\sqrt{x+1} + \sqrt{x}} \, dx$

$= \int \frac{1}{\sqrt{x+1} + \sqrt{x}} \cdot \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x+1} - \sqrt{x}} \, dx$

$= \int \frac{\sqrt{x+1} - \sqrt{x}}{x+1 - x} \, dx$

$= \int \left(\sqrt{x+1} - \sqrt{x}\right) dx$

$= \int \sqrt{x+1} \, dx - \int \sqrt{x} \, dx$

$= \int u^{1/2} \, du - \frac{2}{3} x^{3/2} + C \quad (u = x+1$

$= \boxed{\frac{2}{3}(x+1)^{3/2} - \frac{2}{3} x^{3/2} + C}$

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⑥ $I = \int_0^{\pi/4} \tan^3 x \sec^4 x \, dx$

$= \int_0^{\pi/4} \tan^3 x \left(\tan^2 x + 1\right) \sec^2 x \, dx \qquad u = \tan x \qquad du = \sec^2 x \, dx$

$= \int_0^1 u^3 (u^2 + 1) \, du = \int_0^1 (u^5 + u^3) \, du$

$= \frac{1}{6} u^6 + \frac{1}{4} u^4 \Big|_0^1 = \frac{1}{6} + \frac{1}{4} = \boxed{\frac{5}{12}}$

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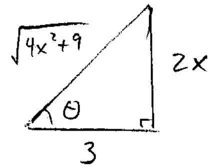
⑦ See solutions for # 4 on Homework 6.

(8) $$I = \int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$$



$x = \frac{3}{2}\tan\theta$

$dx = \frac{3}{2}\sec^2\theta\,d\theta$

$\sqrt{4x^2+9} = 3\sec\theta$

$$= \int_{x=0}^{x=\frac{3\sqrt{3}}{2}} \frac{\left(\frac{3}{2}\tan\theta\right)^3}{(3\sec\theta)^3}\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \frac{3}{16}\int_{x=0}^{x=\frac{3\sqrt{3}}{2}} \frac{\tan^3\theta}{\sec\theta}\,d\theta$$

$$= \frac{3}{16}\int_{x=0}^{x=\frac{3\sqrt{3}}{2}} \frac{\tan^2\theta}{\sec^2\theta}\cdot\sec\theta\tan\theta\,d\theta$$

$$= \frac{3}{16}\int_{x=0}^{x=\frac{3\sqrt{3}}{2}} \frac{\sec^2\theta-1}{\sec^2\theta}\cdot\sec\theta\tan\theta\,d\theta$$

$u = \sec\theta \qquad du = \sec\theta\tan\theta\,d\theta$

(There are other ways to do this, for example, write everything in sines and cosines)

$$= \frac{3}{16}\int_{x=0}^{x=\frac{3\sqrt{3}}{2}} \frac{u^2-1}{u^2}\,du$$

$$= \frac{3}{16}\int_{x=0}^{x=\frac{3\sqrt{3}}{2}} \left(1-\frac{1}{u^2}\right)du$$

$$= \frac{3}{16}\left(u+\frac{1}{u}\right)\Bigg|_{x=0}^{x=\frac{3\sqrt{3}}{2}}$$

$$= \frac{3}{16}\left(\sec\theta+\cos\theta\right)\Bigg|_{x=0}^{x=\frac{3\sqrt{3}}{2}}$$

$$= \frac{3}{16}\left(\frac{\sqrt{4x^2+9}}{3}+\frac{3}{\sqrt{4x^2+9}}\right)\Bigg|_{x=0}^{x=\frac{3\sqrt{3}}{2}} \qquad \left(\left(\frac{3\sqrt{3}}{2}\right)^2 = \frac{9\cdot 3}{4} = \frac{27}{4}\right)$$

$$= \frac{3}{16}\left[\left(2+\frac{1}{2}\right)-\left(1+1\right)\right] = \boxed{\frac{3}{32}}$$

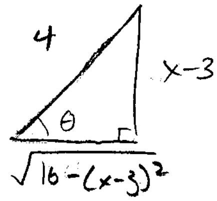
(9) $$I = \int_3^7 \cos^{-1}\left(\frac{\sqrt{7+6x-x^2}}{4}\right) dx$$

Complete the Square

$$-x^2+6x+7 = -(x^2-6x+9)+7+9$$

$$= -(x-3)^2+16$$

$$= \int_3^7 \cos^{-1}\left(\frac{\sqrt{16-(x-3)^2}}{4}\right) dx$$



$$x-3 = 4\sin\theta$$

$$dx = 4\cos\theta \, d\theta$$

(*) $$= \int_{x=3}^{x=7} \theta \cdot 4\cos\theta \, d\theta$$

$u = \theta \qquad dv = \cos\theta \, d\theta$

$du = d\theta \qquad v = \sin\theta$

$$= 4\theta\sin\theta \Big|_{x=3}^{x=7} - 4\int_{x=3}^{x=7} \sin\theta \, d\theta$$

$$= 4\theta\sin\theta + 4\cos\theta \Big|_{x=3}^{x=7}$$

$$= 4\sin^{-1}\left(\frac{x-3}{4}\right)\frac{x-3}{4} + 4\cdot\frac{\sqrt{16-(x-3)^2}}{4} \Big|_{x=3}^{x=7}$$

$$= (x-3)\sin^{-1}\left(\frac{x-3}{4}\right) + \sqrt{16-(x-3)^2} \Big|_{x=3}^{x=7}$$

$$= \left(4\cdot\frac{\pi}{2} + 0\right) - (0+4)$$

$$= \boxed{4\left(\frac{\pi}{2}-1\right)}$$

(10) See from (*) on in # 9.

(11) $I = \int \frac{\sqrt{t+9}}{t}\,dt$ $\quad u = t+9 \quad t = u-9$

$du = dt$

$= \int \frac{\sqrt{u}}{u-9}\,du$ $\quad w = \sqrt{u} \quad u = w^2$

$du = 2w\,dw$

$= \int \frac{w}{w^2-9}\, 2w\,dw$

$= 2\int \frac{w^2}{w^2-9}\,dw$

$= 2\int \left(1 + \frac{9}{w^2-9}\right)dw$

$= 2w + 2\int\left(\frac{-1/2}{w+3} + \frac{1/2}{w-3}\right)dw$

$= 2w + 2\cdot\left(-\frac{1}{2}\ln|w+3| + \frac{1}{2}\ln|w-3|\right) + C$

$= 2w + \ln\frac{|w-3|}{|w+3|} + C$

$= 2\sqrt{u} - \ln\left|\frac{\sqrt{u}-3}{\sqrt{u}+3}\right| + C$

$= \boxed{2\sqrt{t+9} - \ln\left|\frac{\sqrt{t+9}-3}{\sqrt{t+9}+3}\right| + C}$

(12) $$I = \int_1^9 \frac{1}{\sqrt[3]{x-9}}\,dx$$

$$= \lim_{t\to 9^-} \int_1^t (x-9)^{-1/3}\,dx$$

$$= \lim_{t\to 9^-} \frac{3}{2}(x-9)^{2/3}\Big|_1^t$$

$$= \lim_{t\to 9^-} \frac{3}{2}(t-9)^{2/3} - \frac{3}{2}(-8)^{2/3}$$

$$= \frac{3}{2}(0)^{2/3} - \frac{3}{2}(-2)^2 = 0 - \frac{3}{2}(4) = \boxed{-6}$$

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(13) $$I = \int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \ln x \cdot \frac{1}{x^2}\,dx$$

$u = \ln x \qquad dv = \frac{1}{x^2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$

$$= \lim_{t\to\infty} \left(-\frac{\ln x}{x}\Big|_1^t + \int_1^t \frac{1}{x^2}\,dx \right)$$

$$= \lim_{t\to\infty} \left(\left[-\frac{\ln x}{x} - \frac{1}{x} \right]_1^t \right)$$

$$= \lim_{t\to\infty} \left[\left(-\frac{\ln t}{t} - \frac{1}{t} \right) - \left(-\frac{\ln 1}{1} - 1 \right) \right]$$

$$= \underbrace{\lim_{t\to\infty} \left(-\frac{\ln t}{t} \right)}_{\text{Indeterminate}} - 0 + 0 + 1$$

$$\overset{L.H.}{=} \lim_{t\to\infty} \left(-\frac{\frac{1}{t}}{1} \right) + 1$$

$$= 0 + 1 = \boxed{1}$$

(14) $I = \int_0^4 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^4 \frac{1}{(x-1)^2}\,dx$

$$= \lim_{t\to 1^-} \int_0^t \frac{1}{(x-1)^2}\,dx + \lim_{s\to 1^+} \int_s^4 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{t\to 1^-} -\frac{1}{x-1}\Big]_0^t + \lim_{s\to 1^+} \frac{-1}{x-1}\Big]_s^4$$

$$= \lim_{t\to 1^-} \frac{-1}{t-1} - \frac{-1}{-1} + \lim_{s\to 1^+} \frac{-1}{3} - \frac{-1}{s-1}$$

But $\lim_{t\to 1^-} \frac{-1}{t-1} = \infty$, so $\int_0^4 \frac{1}{(x-1)^2}\,dx$ does not converge.

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(15) $I = \int_{-\infty}^{\infty} x^2 e^{-x^3}\,dx = \int_{-\infty}^{0} x^2 e^{-x^3}\,dx + \int_0^{\infty} x^2 e^{-x^3}\,dx$

$$= \lim_{t\to -\infty} \int_t^0 x^2 e^{-x^3}\,dx + \lim_{s\to\infty} \int_0^s x^2 e^{-x^3}\,dx \qquad \begin{aligned} u &= -x^3 \\ du &= -3x^2\,dx \end{aligned}$$

$$= \lim_{t\to -\infty} -\frac{1}{3}\int_{-t^3}^{0} e^u\,du + \lim_{s\to\infty} -\frac{1}{3}\int_0^{-s^3} e^u\,du$$

$$= \lim_{t\to -\infty} -\frac{1}{3} e^u\Big|_{-t^3}^{0} + \lim_{s\to\infty} -\frac{1}{3} e^u\Big|_0^{-s^3}$$

$$= \lim_{t\to -\infty} -\frac{1}{3}\left(e^0 - e^{-t^3}\right) + \lim_{s\to\infty} -\frac{1}{3}\left(e^{-s^3} - e^0\right)$$

But $\lim_{t\to -\infty} e^{-t^3} = \infty$, so $\int_{-\infty}^{\infty} x^2 e^{-x^3}\,dx$ does not converge.

(16) $f(x,y) = \ln(3x + 5y)$

$$f_x(x,y) = \frac{1}{3x+5y} \cdot 3 = \frac{3}{3x+5y}$$

$$f_y(x,y) = \frac{1}{3x+5y} \cdot 5 = \frac{5}{3x+5y}$$

$$f_{xx}(x,y) = 3\cdot\left(-(3x+5y)^{-2}\cdot 3\right) = \frac{-9}{(3x+5y)^2}$$

$$f_{xy}(x,y) = 3\cdot\left(-(3x+5y)^{-2}\cdot 5\right) = \frac{-15}{(3x+5y)^2}$$

$$f_{yx}(x,y) = 5\cdot\left(-(3x+5y)^{-2}\cdot 3\right) = \frac{-15}{(3x+5y)^2}$$

$$f_{yy}(x,y) = 5\cdot\left(-(3x+5y)^{-2}\cdot 5\right) = \frac{-25}{(3x+5y)^2}$$

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(17) $g(x,t) = \arctan(x\sqrt{t})$

$$\frac{\partial g}{\partial x} = \frac{1}{1+(x\sqrt{t})^2} \cdot \sqrt{t} = \frac{\sqrt{t}}{1+x^2t}$$

$$\frac{\partial g}{\partial t} = \frac{1}{1+(x\sqrt{t})^2} \cdot \frac{x}{2\sqrt{t}} = \frac{x}{2\sqrt{t}\,(1+x^2t)}$$

(18) $I = \iint_D e^{x/y}\, dA \qquad D = \{(x,y) : 1 \le y \le 2,\ y \le x \le y^3\}$

$$= \int_{y=1}^{y=2} \left(\int_{x=y}^{x=y^3} e^{x/y}\, dx \right) dy$$

$$= \int_{y=1}^{y=2} y e^{x/y} \Big]_{x=y}^{x=y^3} dy$$

$$= \int_1^2 y\left(e^{y^2} - e^1\right) dy$$

$$= \int_1^2 y e^{y^2}\, dy - \int_1^2 e y\, dy$$

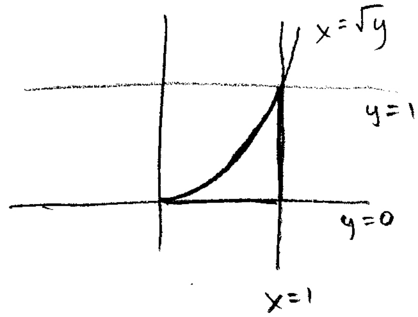
$$= \frac{1}{2} e^{y^2} \Big|_1^2 - \frac{1}{2} e y^2 \Big|_1^2$$

$$= \frac{1}{2} e^4 - \frac{1}{2} e - \frac{1}{2} e \cdot 4 + \frac{1}{2} e$$

$$= \frac{1}{2} e^4 - 2e$$

$$= \boxed{\frac{1}{2} e\left(e^3 - 4\right)}$$

(19) $I = \int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy$



$= \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx$

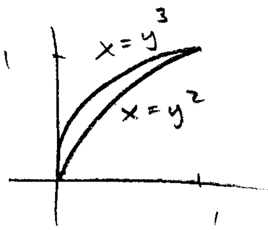
$= \int_0^1 \sqrt{x^3+1} \cdot y \Big|_0^{x^2} dx$

$= \int_0^1 \sqrt{x^3+1} \cdot x^2\, dx \qquad u = x^3+1 \qquad du = 3x^2 dx$

$= \frac{1}{3}\int_1^2 \sqrt{u}\, du = \frac{1}{3} \cdot \frac{2}{3} u^{3/2} \Big|_1^2$

$= \frac{2}{9}\left(2^{3/2} - 1\right) = \boxed{\frac{2}{9}\left(2\sqrt{2} - 1\right)}$

(20) $z = 2x + y^2 \qquad x = y^2, \quad x = y^3$



$$V = \int_{y=0}^{y=1} \int_{x=y^2}^{x=y^3} (2x + y^2)\, dx \; dy$$

$$= \int_{y=0}^{y=1} (x^2 + y^2 x)\Big|_{x=y^2}^{x=y^3} dy$$

$$= \int_{y=0}^{y=1} (y^6 + y^5) - (y^4 + y^4)\; dy$$

$$= \int_0^1 (y^6 + y^5 - 2y^4)\, dy$$

$$= \frac{1}{7}y^7 + \frac{1}{6}y^6 - \frac{2}{5}y^5 \Big|_0^1 = \frac{1}{7} + \frac{1}{6} - \frac{2}{5} - 0$$

$$= \boxed{\frac{-19}{210}}$$

(21) $$\int_1^2 \int_0^{\ln x} f(x,y)\, dy \; dx$$

$$= \int_0^{\ln 2} \int_{e^y}^{2} f(x,y)\, dx\, dy$$