Open book decompositions on bundles

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Open book decompositions and contact structures

Theorem. *(Giroux) On a 3-manifold M: Open books/ (positive) Hopf stabilization $\sim$ Contact structures / isotopy.*

- There is a compatibility condition relating an open book $\mathfrak{ob}$ and a contact structure ξ.
- (Thurston-Winkelnkemper) For any open book, there is a compatible contact structure.
- Any two compatible contact structures are isotopic.
- (Giroux) if $\mathfrak{ob}_1$ $\mathfrak{ob}_2$ are both compatible with ξ, then they are related by a series of positive Hopf stabilizations: $\mathfrak{ob}_1 \natural H^+ \cdots \natural H^+ \simeq \mathfrak{ob}_2 \natural H^+ \cdots \natural H^+$

The canonical example

There are two (mostly equivalent) ways of decribing open books: embedded as fibered links and as abstract bundles (Σ, ϕ), where Σ is a bounded surface and $\phi \in Aut^+(\Sigma)$ (or Map^+).

Example: (S^3, ξ_{std}): the (unique) tight contact structure on S^3 is compatible with the open book given by the unknot (D^2, id).

The question of genus

Genus tells us interesting things about geometry:
If ξ is compatible with a genus 0 open book

- (Ozsvath-Szabo) The HF^+ contact invariant is reducible. "it lies in $HF^+_{red} \subseteq HF^+$"
- (Etnyre) Any (weak) symplectic filling must be negative definite, and diagonalizeable if unipotent.
- (Etnyre) Any overtwisted contact structure is genus 0.

Contact structures on torus bundles

Contact structures on T^3 were classified by Kanda ('97). Up to isomorphism they are determined by an invariant (the *Giroux torsion* $\tau \in \mathbb{Z}^{\geq 0}$). All are covers of (T^3, ξ_0) where $\tau = 0$.

- (T^3, ξ_0) is Stein fillable, realized as the contact boundary of a neighborhood of the Clifford torus in $\mathbb{C}^2$.
- (T^3, ξ_i) is not strongly fillable if $i > 0$ (Eliashberg).
- (T^3, ξ_i) is weakly filled by the indefinite $T^2 \times D^2$.
- These cannot be genus 0!
- We will see that these are genus 1.

What does Stein fillability mean for us?

Open books arise naturally as the boundaries of Lefschetz fibrations over the disk, with bounded fiber. These have natural Stein (symplectic) structures. These restrict compatibly to the boundary.

Theorem. *A contact structure is Stein fillable if and only if it is compatible with an open book* (Σ, ϕ)*, where* $\phi \in Dehn^+ \subset Map^+$ *is in the monoid generated by* positive dehn twists*.*

Tools for the construction:

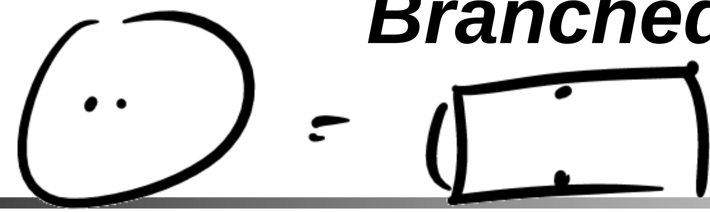
	4-manifolds	3-manifolds
		induced boundary
geometry	Stein str.	contact structure
compatibility	$\updownarrow$	$\updownarrow$
topology	Lefschetz fibr.	open book

Similarly, there are symplectic operations that can be done relatively and compatibly.

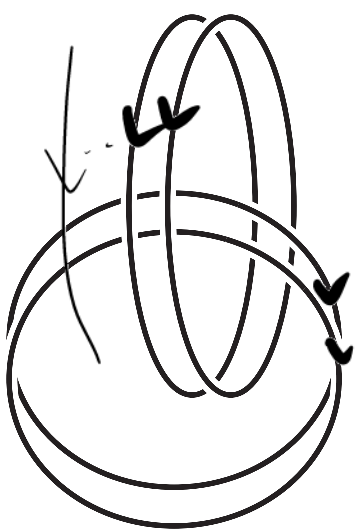
Branched Covers

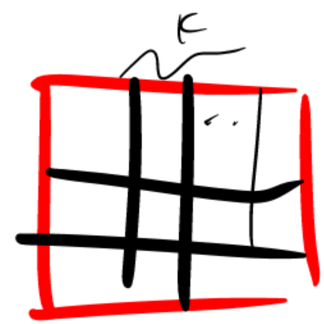
- Stein structures lift under branched covers along analytic hypersurfaces.
- contact structures lift under branched covers along knots (links) transverse to the contact planes.
- open book decompositions lift under branched covers along knots (links) that are everywhere transverse to the fibers.
- these branched covers can be done simultaneously while preserving compatibility.

Branched cover construction of $T^2 \times D^2$



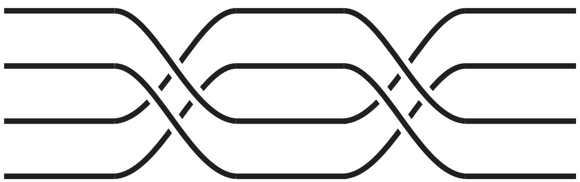
The natural Stein structure on $T^2 \times D^2$ arising as the neighborhood of the Clifford torus is just the product $(\mathbb{A} \times \mathbb{A}, \omega \oplus \omega)$. We can build this by taking the (2×2)-fold cover of $(D^2 \times D^2, \omega \oplus \omega)$ along four product disks $D^2 \times p_i \cup q_i \times D^2$ $i = 1, 2$. This is an analytic surface with boundary the untwisted double of the positive hopf link:

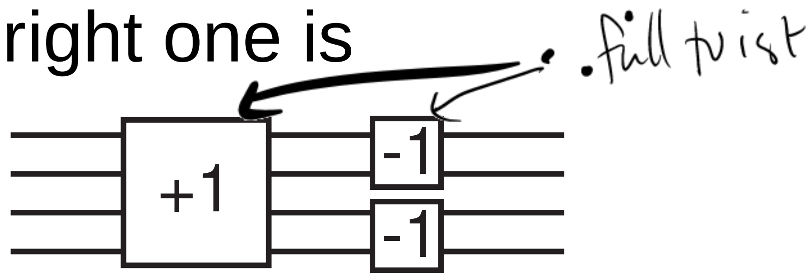




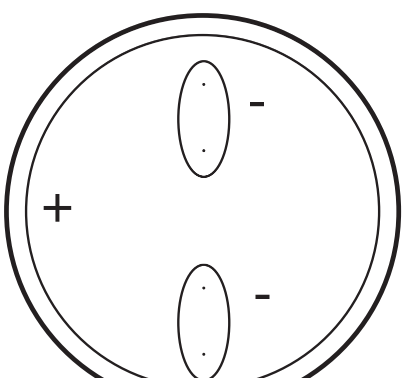
The perturbation

As presented, though, the natural open book on S^3 is given by an unknot parallel to one component of the branch locus. Perturbing to be transverse means turing L into a braid. The right one is



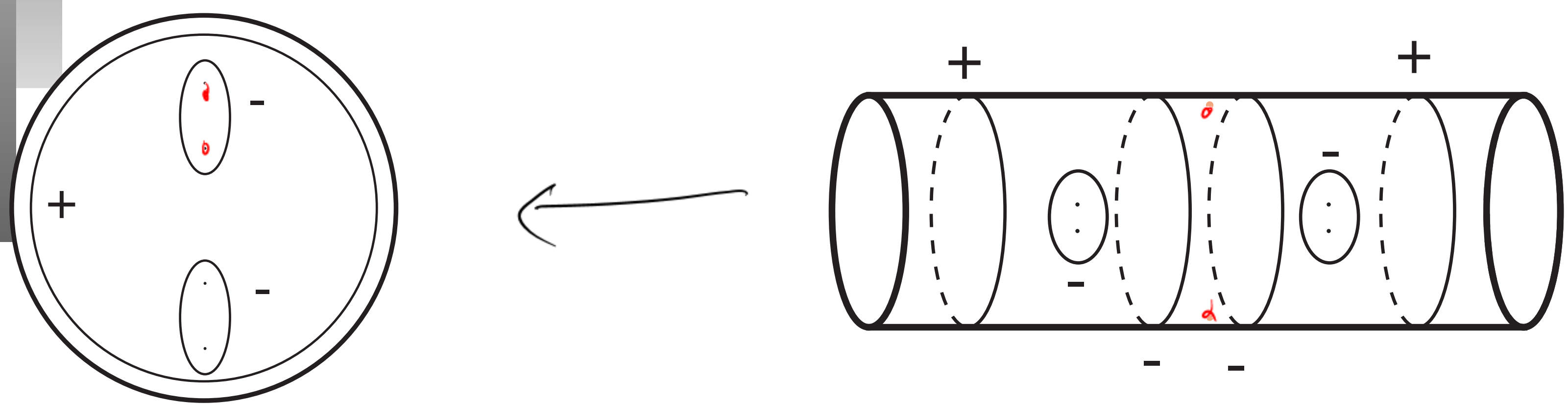


This is a pure braid, though, which means lifting the monodromy is easy. Looking at the action on the disk with marked points we have

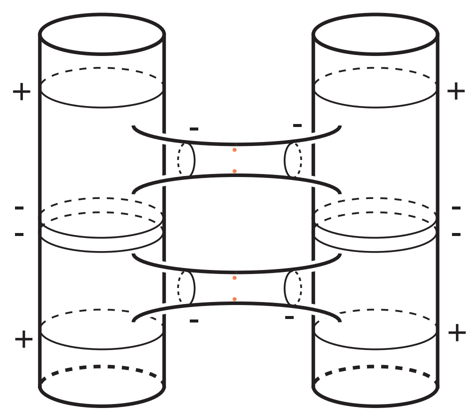


Lifting the monodromy

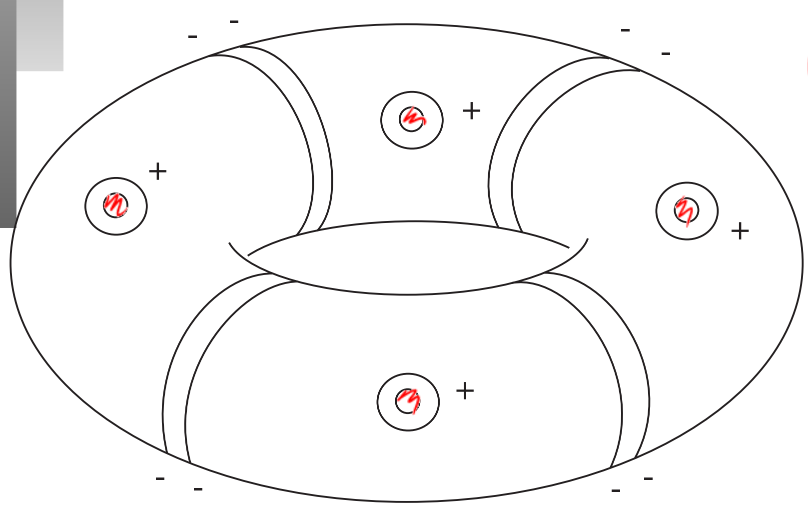
We do the branched cover in two steps.

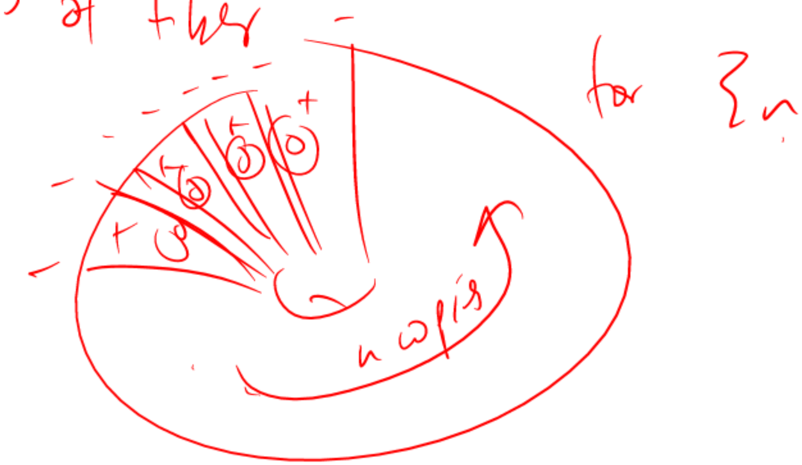


and then



Rewriting the diagram, we get (T^3, ξ_0)

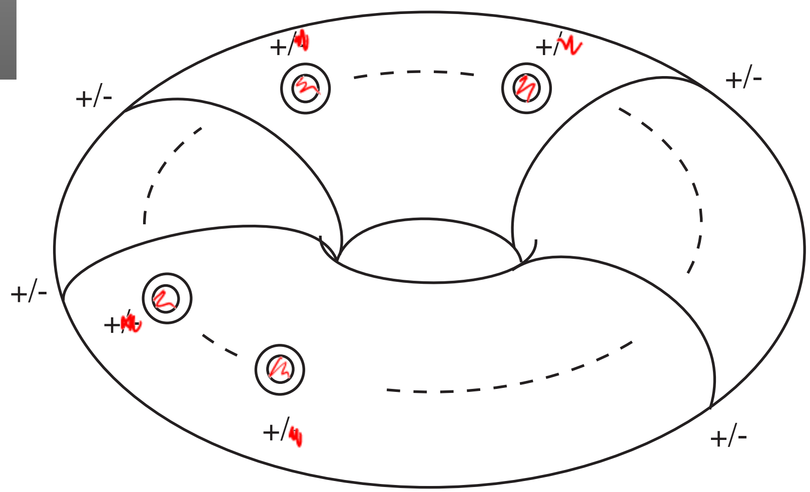


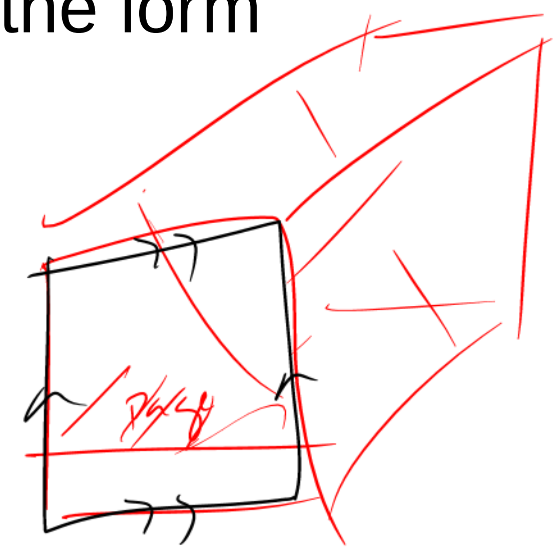


Things become interesting when you look at this picture embedded (but we don't have time for that here). In particular, one can show the monodromy is in $Dehn^+$. This is Stein fillable.

The general picture

For a general torus bundle, T_A and a contact structure ξ_i with torsion $\tau = i$ (as discussed before), (T_A, ξ_i) is compatible with an open book of the form





Patterns

- All (most) tight contact structures on torus bundles are genus 1.
- There are genus $2g$ $(+1)$ open books on the tight contact structures on circle bundles over a genus g surface.
- Gives a (mostly) complete classification.
- Genus 1 gives (hopefully) interesting monodromy restrictions: We understand $Dehn^+(T^3)$ and we understand elliptic fibrations.