RESEARCH STATEMENT - JONATHAN D. WILLIAMS

1. INTRODUCTION

One of the more intriguing aspects of low-dimensional topology concerns smooth structures in dimension 4. For a closed topological manifold, there are only finitely many (perhaps zero) non-diffeomorphic smooth representatives or "smooth structures," except in dimension 4. In that case, there are many examples with infinitely many smooth structures, and there is no proposed classification scheme even for a single homeomorphism class. The Seiberg-Witten invariant has been the most useful tool for distinguishing 4-manifolds which are homeomorphic but not diffeomorphic, but this has essentially always required at least one of them to admit a symplectic structure, that is a closed, nondegenerate 2-form. I am interested in the effort to generalize the use and understanding of smooth invariants like Seiberg-Witten to the nonsymplectic case, and in related efforts to define possibly new smooth invariants. These pursuits involve broken Lefschetz fibrations; these are smooth maps from 4-manifolds to surfaces that satisfy various restrictions on their critical loci.

2. PAST RESEARCH

2.1. **Context.** An obvious way to try to understand the smooth phenomena of 4-manifolds is to better understand the Seiberg-Witten invariant, which is defined by an elliptic pair of PDE involving a connection and a spin^c structure on the tangent space of the manifold. Progress toward a more geometric, hands-on understanding began in 1996 when Taubes showed that on a symplectic manifold solutions to the Seiberg-Witten equations correspond to pseudoholomorphic curves [T]. This made it plausible that there were purely geometric proofs of previous results gained using the Seiberg-Witten equations, and toward this end there appeared a *standard surface count* due to Donaldson and Smith [DS], essentially a Gromov invariant counting holomorphic sections of a fiber bundle associated to a symplectic Lefschetz fibration. There is an interesting progression here: The tremendously fruitful handlebody theory (and its package of classical invariants) came from studying the critical loci of real-valued Morse functions, and Lefschetz fibrations are a complex version of Morse functions (their critical points are quadratic in suitable complex coordinates). The chain complexes of Floer theory can be viewed as complex versions of those appearing in Morse theory, and the limits of their utility and generality are a highly active area of research.

Three years after the appearance of [DS], Usher showed that a version of the standard surface count (defined as a symplectic invariant) is equivalent to the Seiberg-Witten invariant of the underlying smooth 4-manifold. The result was a geometric understanding of Seiberg-Witten theory coming from Lefschetz fibrations. In [ADK], Auroux, Donaldson and Katzarkov took an initial step toward extending this approach toward the nonsymplectic setting, showing that, just as symplectic 4-manifolds correspond to Lefschetz fibrations, near-symplectic 4-manifolds (in which the 2-form is allowed to vanish along circles) correspond to objects called *broken Lefschetz fibrations*. These generalize Lefschetz fibrations by allowing an additional kind of critical point, known as an indefinite fold point. The indefinite fold locus forms a 1-dimensional submanifold of the total space of the fibration, and each circle is commonly called a broken singularity. Topologically, the preimage of an arc that passes across the image of a broken singularity (which is generically immersed by the fibration map) is a cobordism between the endpoint fibers obtained by adding a 1- or 2-handle to (fiber) $\times I$, equipped with the fibration structure that comes from an index 1 or 2 Morse critical point (a definite fold would give index 0 or 3).

The next step in this program is to extend the analogy between the two fibration structures to find an appropriate generalization of the standard surface count, and the Lagrangian matching theory appearing in [P] is a likely candidate. Here the 4-manifold is decomposed into a union of symplectic Lefschetz fibrations, each of which has an associated bundle whose holomorphic sections are required to satisfy boundary conditions also lifted from the original fibration structure. Another "quilt theory" of Wehrheim and Woodward [WW], which may contain what appears in [P] as a special case, offers another possible repackaging of the Seiberg-Witten invariant using fibration structures as auxiliary data; it is unclear whether a near-symplectic structure is necessary in their case. Currently, it is not known if either theory even presents an invariant of the underlying smooth 4-manifold – a thorough understanding of broken Lefschetz fibrations will be necessary to get an answer to this question, and this has been a primary motivation for my research.

2.2. **Results.** Certainly, the addition of indefinite folds adds a lot of variety, and it is still unknown whether the so-called invariants associated to them depend on the chosen fibration structure. One issue is topological: what broken fibrations are possible on a given 4-manifold? When [P] appeared in 2006, it was not even known which 4-manifolds admit broken Lefschetz fibrations. Soon it was shown that every smooth 4-manifold may be given the structure of a broken Lefschetz fibration ([B2, AK, GK]), and there was a list of moves, coming from singularity theory, that were conjectured to connect any pair of homotopic broken fibration maps [L]. Crucial to the definition of any associated invariant, my first result was that these moves were indeed sufficient for that purpose.

Theorem 1. If two broken Lefschetz fibrations are homotopic, then there exists a homotopy between them which is realized as a sequence of moves from the list appearing in [L].

Usually, broken Lefschetz fibrations are maps to S^2, and the homotopy classes of such maps correspond to framed cobordism classes of surfaces in M by the Pontrjagin-Thom construction. Implicit in the above mentioned existence results is that there is a broken Lefschetz fibration in each homotopy class. For this reason, a further result is necessary for uniqueness when the base of the fibration is S^2. The following result serves both to complete the uniqueness theorem for sphere-valued fibrations, and to single out what may turn out to be a favored class of maps to the disk, as discussed in Section 3.1.

Theorem 2. Given a broken Lefschetz fibration over S^2, the list of moves appearing in [L] and an additional *projection* move generate all possible broken Lefschetz fibration structures on M.

This projection move is a rather simple, explicit way to convert the base from S^2 to D^2, resulting in a map which is "as close as possible" to being an actual broken Lefschetz fibration. The proof of this theorem relies on Theorem 1 because of an argument showing that, once the projection move is applied, such maps are related by the remaining moves.

A further consequence of Theorem 1 is an existence result for certain generic maps which I call *simplified purely wrinkled fibrations*, combining terminology of Baykur and Lekili. These give a novel, essentially combinatorial way to specify smooth 4-manifolds.

Theorem 3. Any smooth closed 4-manifold may be specified by a chain $\{\gamma_i\}_{i\in\mathbb{Z}/k\mathbb{Z}}$ of simple closed curves in an orientable closed surface F such that γ_i transversely intersects γ_{i+1} at a unique point.

The base of one of these maps is a sphere with a distinguished circle that has k cusps, breaking it up into k smooth arcs. Each smooth arc has γ_i as a vanishing cycle within the fiber F, and this sequence of vanishing cycles specifies the northern hemisphere of a broken fibration. Assuming the genus of F is greater than 1, there is a unique way to glue in the fibration over southern hemisphere. This assumption on fiber genus can be easily satisfied using a stabilization procedure discussed in Section 3.2.

3. PRESENT AND FUTURE DIRECTIONS

The broad goal to use broken fibrations to prove theorems about smooth 4-manifolds is a long-term effort still in its early stages. An important step is to prove that there really are smooth invariants, and below are two approaches to this problem that stem directly from my previous work.

3.1. **Holomorphic Quilts.** Wehrheim and Woodward have begun to develop a general Floer theory that specializes to use the fibration structure coming from a generic map $f : M^4 \to N^2$. For broken fibrations, one of their goals is to use the fibration structure to create a moduli space of holomorphic curves with Lagrangian boundary conditions, and from this to extract a Floer cohomological invariant depending only on the smooth 4-manifold which is the total space of the fibration. Quilts come from a bottom-up approach, starting with surfaces and using TQFT theory to guide the definition of higher-dimensional invariants. The Lagrangian matching theory has a top-down construction that begins with a 4-manifold and directly constructs a moduli space meant to mimic the Seiberg-Witten invariant, using a geometric construction that turns out to satisfy various functorial attributes such as the existence of a fiber sum theorem and a relative invariant. Currently, it is not known whether either approach yields a smooth invariant; for example, there are no results that relate the objects resulting from a pair of maps which are not homotopic. The proof of Theorem 2 shows that, once a pair of fibrations $M \to S^2$ have been projected, they can be connected by a sequence of the other moves, each of which is realized by a homotopy. In most cases, there is work that indicates it will be possible to show that these other moves do not change the Floer-theoretic invariants. It would be interesting to combine the machinery of quilts with the boundary conditions of [P] in a Floer theory for projected maps, a project that would require a nontrivial extension of both. The prospect of even defining a moduli space for general maps to the disk seems daunting for numerous reasons; however, a projected fibration retains enough of the original structure that, in that case, it may be possible to realize a moduli space as a moderate generalization of what has been done for the sphere. Establishing such a construction would be a promising step toward proving the existence of smooth Floer-theoretic invariants.

3.2. **The topological approach.** Simplified purely wrinkled fibrations and their associated surfaces with chains of simple closed curves offer a new angle from which to address the theory of smooth 4-manifolds, distinct even from the relatively recent appearance of broken fibrations. They offer a combinatorial framework much like Heegaard diagrams for 3-manifolds, with a somewhat similar stabilization coming from a special case of a procedure appearing in Figures 5 of [B2] and 11 of [L], and handleslides corresponding to Hurwitz moves on Lefschetz fibrations. Investigating just how far this similarity extends with respect to Heegaard-Floer theory would be a worthy project. For a fixed homotopy class of such maps, there is a likely uniqueness result stating, essentially, that their associated surface diagrams can be related by stabilizations and Hurwitz moves. It is essentially unexplored territory, and until the appearance of these diagrams, there has not been a plausible inroad for defining a combinatorial smooth invariant of 4-manifolds. Finally, the proof of Theorem 2 stops short of an h-principle for maps to the disk with with a single definite circle (that is, those maps to the disk that most closely approximate broken fibrations) mainly because a shorter, cleaner argument was available for the uniqueness result 2. With sufficient motivation coming from developments in the theories above, it would be fruitful and probably not too difficult to prove such a result. Thus there are several opportunities for the topologically minded to pursue the broad goal stated above.

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