CHARACTER VARIETIES: REID

1. Setting up the varieties

1.1. **Jan. 20: Intro.** Γ f.g., f.p. group, G a Lie group, either $\mathbb{R}, \mathbb{C}$, or $\mathbb{Q}_p$. eg. (P)$\mathrm{SL}_2(X)$ for $X \in \{\mathbb{R}, \mathbb{C}, \mathbb{Q}_p\}$. A **representation** is a homomorphism $\rho : \Gamma \to G$.
Remarks.

(1) $\forall\, \Gamma, G$ there exists $\rho_{triv} : \Gamma \to G$ with $\gamma \mapsto 1\ \forall\ \gamma \in \Gamma$.
(2) $\rho : \Gamma \to G$ nontrivial $\implies$ potentially lots of representations (eg. can conjugate). Given $g \in G$, this defines $\varphi_g : G \to G$ so we get $\varphi_g \circ \rho : \Gamma \to G$.
(3) $G \leq \mathrm{GL}_n(\mathbb{C}) \implies \rho(\gamma)$ is a matrix $\forall\ \gamma \in \Gamma$.
   eg1: $\Gamma =< a, b > \implies$ can define ρ by $a \mapsto g_a$ and $b \mapsto g_b$.
   eg2: more generally, if Γ has relations, must find a suitable representation...
(4) $\rho : \Gamma \to G$ is **faithful** if ρ is 1-1. Geometry affords faithful representations in certain settings.
   eg: $M^n = \mathbb{H}^n/\Gamma$ is a hyperbolic manifold implies that we have an induced isomorphism $\rho : \pi_1(M^n) \to \Gamma$.

Call Γ a **linear group** if Γ admits a faithful representation into some $\mathrm{GL}_n(\mathbb{C})$.

- it is interesting to determine which groups are linear, eg. mapping class groups, Haken 3-manifold groups
- even weaker is determining whether a group is (non)-trivial
  eg: general triangle group: $< a, b | a^n = b^m = w^l = 1 >$ where w is a reduced word in a, b
  eg2: Can $G = C_{p_1} * C_{p_2} * \ldots * C_{p_r}, r \geq 2, p_i$ distinct primes be killed, i.e. $G/ < w >= 1$?

f.g. linear groups are **residually finite**, so they have lots of finite quotients.
One can form $\mathrm{Hom}(\Gamma, G)$; eg. moduli spaces, deformation spaces.
  eg: $\Gamma = \pi_1(\Sigma_g), g \geq 2, G = PSL_2(\mathbb{R})$ (or $\mathbb{C}$)

Definition 1. *Let $\rho : \Gamma \to \mathrm{GL}_n(K), \mathrm{char}(K) = 0$. Define $\chi_p : \Gamma \to K$ by $\chi_p(\gamma) = \mathrm{Tr}(\rho(\gamma))$.* ***note:*** *$\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \implies \chi_\rho$ is constant on conjugacy classes.*

Let
$$\rho : \Gamma \to \mathrm{GL}_n(K)$$
where K is a field with characteristic zero. If $V = K^n$ then $\mathrm{GL}_n(K)$ acts on V by $v \mapsto A\dot{v}$.

Definition 2. *ρ is **irreducible** if the only irreducible subspaces of V under the action of $\rho(\Gamma)$ are $\{0\}$ and V. Otherwise ρ is **reducible**.*

example: $G = \mathrm{SL}_2(\mathbb{C})$.

$$\mathbf{B} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

note: B leaves invariant the 1-dimensional subspace generated by $(*, 0)^T$.

If $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ has $\rho(\Gamma) \subseteq B$ then ρ is reducible. Indeed, $\rho(\Gamma)$ is conjugate into B implies that ρ is reducible. Furthermore, the above example yields that $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ reducible implies that $\rho(\Gamma)$ can be conjugated into B. In sum, we need to understand irreducible representations in $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$;

Let $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ be a reducible representation. Conjugate so that $\rho(\Gamma) \subset B$, and consider a commutator $c \in [\Gamma, \Gamma]$. Then

$$\begin{aligned} \rho(c) &= \begin{pmatrix} \lambda & s \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \mu & t \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} \lambda^{-1} & -s \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \mu^{-1} & -t \\ 0 & \mu \end{pmatrix} \\ &= \begin{pmatrix} \lambda\mu & \lambda t + s\mu^{-1} \\ 0 & \lambda^{-1}\mu^{-1} \end{pmatrix} \begin{pmatrix} \lambda^{-1}\mu^{-1} & -t\lambda^{-1} - s\mu \\ 0 & \lambda\mu \end{pmatrix} \\ &= \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Hence, $\chi_\rho(c) = 2$ if $c \in [\Gamma, \Gamma]$.

Lemma 1.1. *$\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ is reducible iff $\chi_\rho(c) = 2$ for every $c \in [\Gamma, \Gamma]$.*

Proof. $\Rightarrow$: done above.
$\Leftarrow$: First suppose that Γ is abelian, $\rho : \Gamma \to \mathrm{GL}_n(\mathbb{C}), n \geq 2$ is a representation. Let $\rho(\gamma) \neq 1$ and consider $X =$ the eigenspace of $\rho(\gamma)$. If $X = \mathbb{C}^n$, then $\rho(\delta) = \lambda_\delta I \ \forall\ \delta \in \Gamma \to\leftarrow$. Otherwise, it is enough to show that $\rho(\delta)x \in X \ \forall\ \delta \in \Gamma, x \in X$. Thus ρ is reducible, so we're done in this case.

Now assume that Γ is not abelian, so there exists $c_0 \in [\Gamma, \Gamma] \ni: \rho(c_0) \neq 1$. This means that $\rho(c_0)$ is $\mathrm{SL}_2(\mathbb{C})$ conjugate to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ and hence there is a 1-dim'l invariant subspace L. Suppose there is a $c \in [\Gamma, \Gamma] \ni: \rho(c)L \neq L$. But $\chi_\rho(c) = 2)$ and $\rho(c) \neq 1$, so as above, there is a 1-dim'l invariant subspace.

By conjugating (linear algebra or Möbuis transformation), we can assume that

$$\rho(c_0) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \rho(c) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, st \neq 0$$

But $\chi_\rho(c_0 c) = 2 + st \neq 2 \to\leftarrow$.
claim: $\rho(\Gamma, \Gamma)$ has a unique 1-dim'l invariant subspace L.

If true, then we're done since $\gamma \in \Gamma, c \in [\Gamma, \Gamma] \implies \rho(\gamma)^{-1}\rho(c)\rho(\gamma)L = \rho(c')L = L$ where $c' \in [\Gamma, \Gamma]$, so $\rho(c)\rho(\gamma)L = \rho(\gamma)L \therefore \rho(\gamma)L = L$ by uniqueness. $\square$

note: characters do not determine representations up to conjugation. **example:** $\rho_{triv} : \mathbb{Z} \to I, \rho : \mathbb{Z} \to \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$

$\chi_{\rho_{triv}} = \chi_\rho \ \forall\ \gamma \in \mathbb{Z}$. This is a feature of reducible representations.

1.2. **Jan. 22: Irreducible representations.** We have now an addendum to Lemma 1.1:

$$\begin{aligned} \rho : \Gamma \to \mathrm{SL}_2(\mathbb{C}) \text{ is reducible } & \text{ iff } \rho(\Gamma) \text{ is conjugate into a group of upper triangular matrices.} \\ & \text{ iff } \text{all the elements of } \rho(\Gamma) \text{ share a common eigenvector.} \\ & \text{ iff } \chi_\rho(c) = 2 \text{ for every } c \in [\Gamma, \Gamma]. \end{aligned}$$

Characters of reducible representations are also characters of abelian representations. If $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ is reducible, then conjugate so that $\rho(\Gamma)$ is upper triangular. Now, if

$$\rho(\gamma) = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$$

then define an abelian representation with the same character by

$$\rho'(\gamma) = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$$

ρ' turns out to be a homomorphism.

Theorem 1.2. *Let ρ, ρ' be representations of Γ into $\mathrm{SL}_2(\mathbb{C})$. Assume that ρ is irreducible and $\chi_\rho = \chi_{\rho'}$. Then ρ and ρ' are $\mathrm{SL}_2(\mathbb{C})$ conjugate.*

Proof. ρ irreducible representation implies that there exists $g \in \Gamma \ni$: $\rho(g) \neq \pm I$, so by Lemma 1.3 there exists an h that satisfies the conclusion of that Lemma. Write $G = \langle g, h\rangle$ and note that $\rho|G$ is an irreducible representation by the Lemma and thus as above it follows that $\rho'|G$ is an irreducible representation , since $\chi_\rho(\gamma) = \chi_{\rho'}(\gamma)\forall\gamma \in \Gamma \implies \chi_\rho = \chi_{\rho'}(\gamma)\forall\gamma \in G$.

Conjugate $\rho(\Gamma), \rho'(\Gamma) \ni$:

$$\rho(h) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \rho'(h)$$

with $\lambda \neq \pm 1$.

note: if $\rho(g) = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ (after above conjugation), then we can conjugate $\rho(\Gamma)$ by $\begin{pmatrix} b_0^{-1/2} & 0 \\ 0 & b_0^{1/2} \end{pmatrix}$ so that $\rho(g) = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$. Similarly for ρ': $\rho'(g) = \begin{pmatrix} a' & 1 \\ c' & d' \end{pmatrix}$. The fact that ρ, ρ' are irreducible representations impliles that $c, c' \neq 0$.

claim: After this normalization, $\rho = \rho'$. **Pf. of claim:** We will prove the claim in four easy steps.

Step 1: Use the fact that $\chi_\rho(\gamma) = \chi_{\rho'}(\gamma)\forall\gamma \in \Gamma$. Thus $a + d = a' + d' \implies a - a' = -(d - d')$.

Step 2: Consider $\chi_\rho(gh) = \chi_{\rho'}(gh)$:

$$\begin{aligned} \rho(gh) &= \rho(g)\rho(h) = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} a\lambda & * \\ * & d\lambda^{-1} \end{pmatrix} \\ \rho'(gh) &= \rho'(g)\rho'(h) = \begin{pmatrix} a' & 1 \\ c' & d' \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} a'\lambda & * \\ * & d'\lambda^{-1} \end{pmatrix} \end{aligned}$$

Therefore $a\lambda + d\lambda^{-1} = a'\lambda + d'\lambda^{-1}$, and thus $a - a' = 0$ since $\lambda^2 \neq 1$.

Hence $d = d'$ and the condition on the determinant yields $c = c'$.

Step 3: Consider $\rho(\gamma) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \rho'(\gamma) = \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}$ for any $\gamma \in \Gamma$. By the character condition, $p + s = p' + s'$ and by considering γh, $p = p'$ and $s = s'$.

Step 4: Consider the above arugment applied to $g\gamma$. Then from above, $ap + r = ap + r'$ and $cq + ds = cq' + ds$, so we're done since $c \neq 0$ by the above note.

$\diamond$

Now we're done with the proof of the theorem since we can pre- and post- compose with conjugate elements. $\square$

The proof above uses the following lemma.

Lemma 1.3. *Let $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ be an irrep. such that if $g \in \Gamma$ where $\rho(g) \neq \pm I$, then there exists an $h \in \Gamma$ where the restriction $\rho|_{\langle g,h \rangle}$ is an irrep. and $\chi_\rho(h) \neq \pm 2$.*

Proof. **claim:** There exists an $h_0 \in \Gamma \ni$: $\rho(g), \rho(h_0)$ have no common eigenvector.
Pf. of claim: (exercise) $\diamond$
If $\chi_\rho(h_0) \neq \pm 2$ then done by claim since ρ is irreducible. Assume that $\chi_\rho(h_0) = \pm 2$. Since $\rho(h_0) \neq \pm 1$ by claim, we may conjugate $\rho(\Gamma) \ni$: $\rho(h_0) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (since $\rho(\Gamma) \subset \mathrm{SL}_2(\mathbb{C}), \rho(h_0) = \begin{pmatrix} 1 & x \\ y & 1 \end{pmatrix} \implies xy = 0$).

To finish, choose large n and consider $h = gh_0^n$. So $\chi_\rho(gh_0^n) = \mathrm{Tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = a + d + cn$. Note that by the claim, $c \neq 0$. Can choose $n \ni$: $\chi_\rho(gh_0^n) \neq \pm 2$. Also note that $\rho(gh_0^n)$ and $\rho(g)$ do not have a common eigenvector; otherwise $\rho(h_0^n)$ and $\rho(g)$ have common eigenvector implies that $\rho(h_0)$ and $\rho(g)$ do as well (since $\rho(h_0), \rho(h_0)^n \subset B$).

Therefore we're done by the addendum to Lemma 1.1. $\square$

Definition 3. *We say that two representations ρ and ρ' are **equivalent** if there exists and inner automorphism $\varphi_g : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})$ with $\rho = \varphi_g \circ \rho'$.*

The theorem shows that if ρ is an irrep. and $\rho \sim \rho' \iff \chi_\rho = \chi_{\rho'}$. Or equivalently, $[\rho]$ is the $\mathrm{SL}_2(\mathbb{C})$ orbit under the action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ by conjugation.

1.3. **Jan. 27: Representation varieties.**

Theorem 1.4. *If* Γ *is a finitely generated group then* $R(\Gamma)$ *is an algebraic set defined over* $\mathbb{Q}$.

Proof. Γ is finitely generated, so fix a generating set $G = \{\gamma_1, \dots, \gamma_n\}$. If $\rho \in R(\Gamma)$, then ρ is determined by $\rho(\gamma_i) = \begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix}, 1 \le i \le n, x_i w_i - y_i z_i = 1$.

As in the examples below, $\rho \leftrightarrow (x_1, \dots, w_n) \in \mathbb{C}^{4n}$.

To determine the subset of $\mathbb{C}^{4n}$, we have:

- determinant conditions
- equations that come from $\rho(R_j(\gamma_1, \dots, \gamma_n)) = I$ for all relators R_j.

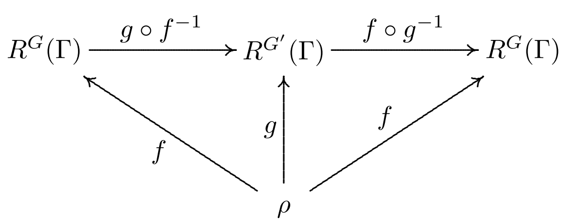
This defines an algebraic subset $R^G(\Gamma) \subset \mathbb{C}^{4n}$.

It remains to show that $R^G(\Gamma) = R^{G'}(\Gamma)$ for another generating set G'.

Suppose that $G' = \{\delta_1, \dots, \delta_m\}$, so we have the algebraic set $R^{G'}(\Gamma) \subset \mathbb{C}^{4m}$.

Since $\{\gamma_i\}$ generates, each $\delta_j = w_j(\gamma_1, \dots, \gamma_n)$. Hence, at the level of representations, each of the x_i', y_i', z_i', w_i' are polys in x_i, y_i, z_i, w_i.

Write $\rho(\delta_i) = \begin{pmatrix} x_i' & y_i' \\ z_i' & w_i' \end{pmatrix}$. These two maps are poly maps that are natural inverses. □



Recall, if $V \subset \mathbb{C}^n$, then V is an **algebraic set** if

$$V = \{(x_1, \dots, x_n) : f_i(x_1, \dots, x_n) = 0 \ \forall \ 1 \le i \le k\}$$

where $f_i \in \mathbb{C}[x_1, \dots, x_n]$.

If $k \subset \mathbb{C}$ is a subfield, then V defined over k if $f_i \in k[x_1, \dots, x_n]$. Note that for any $g \in < f_1, \dots, f_n > \subset \mathbb{C}[x_1, x_n]$, g vanishes on V.

Thus, to an algebraic set V we associate an ideal $I(V)$, the ideal generated by the defining polynomials. Similarly, to an ideal I we associate an algebraic set

$$V(I) = \{(x_1, \dots, x_n) : f(\overline{x}) = 0 \ \ \forall \ \ f \in I\}$$

Furthermore,

$$\sqrt{I} \ = \ \{f : f^m \in I, \ m \ge 1\} \ = \ I(V(I))$$

Remarks.

1. V is irreducible iff $I(V)$ is prime.
2. V may be decomposed into a finite union of irreducibles.
3. If V is defined over $\mathbb{Q}$ its irreducible components need not be. Eg: $p(x) \in \mathbb{Z}[x]$ irreducible and with degree bigger than one.

example: $\Gamma =< a, b >, \rho \in R(\Gamma)$, $\rho(a) = \begin{pmatrix} x_1 & y_1 \\ z_1 & w_1 \end{pmatrix}, \rho(b) = \begin{pmatrix} x_2 & y_2 \\ z_2 & w_2 \end{pmatrix}$. Via the obvious embedding $\mathrm{SL}_2(\mathbb{C}) \hookrightarrow \mathbb{C}^4$, we may identify $\rho \leftrightarrow (x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2) \in \mathbb{C}^8$.

Thus we identify $R(\Gamma) \subset \mathbb{C}^8$ with $\{x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2) : x_i w_i - y_i z_i = 1\}$, so $R(\Gamma)$ is an algebraic set over $\mathbb{Q}$, but now let's focus on irreducible representations .

Remarks.

(1) $\rho : G \to \mathrm{SL}_2(\mathbb{C})$ irreducible representation implies that $\exists \gamma, \delta \in G$ such that we can conjugate ρ so that

$$\rho(\gamma) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix}, \rho(\delta) = \begin{pmatrix} \mu & 0 \\ r & \mu^{-1} \end{pmatrix}$$

with $\lambda, \mu, r \in \mathbb{C}, r \neq 0$.

Since we are concerned with conjugacy classes of representations, we often normalize the representations by conjugating. For two generator groups the following is standard

$$\rho(a) \ = \ \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad \text{and} \qquad \rho(b) \ = \ \begin{pmatrix} \mu & 0 \\ r & \mu^{-1} \end{pmatrix}$$

ρ is an irrep. when $r \neq 0$.

Given an irreducible representation of Γ, we get $(\lambda, \mu, r) \in \mathbb{C}^3$. Conversely, given $(\lambda, \mu, r) \in \mathbb{C}^3$, we get an irreducible representation (when $r \neq 0$).

In fact, given $z_1, z_2, z_3 \in \mathbb{C}^3$, solve for $z_1 = \chi_\rho(a), z_2 = \chi_\rho(b), z_3 = \chi_\rho(ab)$.

Lemma 1.5. *$G =< \gamma_1, \gamma_2 > \subset \mathrm{SL}_2(\mathbb{C})$ then $\forall g \in G, \mathrm{Tr} g = P(\mathrm{Tr}\gamma_1, \mathrm{Tr}\gamma_2, \mathrm{Tr}\gamma_1\gamma_2)$ where $P \in \mathbb{Z}[X, Y, Z]$.*

From lemma, $\forall \gamma \in \Gamma, \chi_\rho(\gamma)$ is determineed by $(\chi_\rho(a), \chi_\rho(b), \chi_\rho(ab)) =: \chi$ so that for $\chi_\rho \in X(\Gamma)$ (where ρ is irreducible representation), this implies that χ_ρ is completely determined by χ.

example:

$$\Gamma = \pi_1(S^3 \setminus K) =< x_1, x_2 : [x_1^{-1}, x_2] x_1 [x_1^{-1}, x_2]^{-1} = x_2 >$$

. If $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ is a representation, then it is determined by $\rho(x_1), \rho(x_2)$:

$$\rho(x_i) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, a_i d_i - b_i c_i = 1$$

Via the embedding above, $\mathrm{SL}_2(\mathbb{C}) \hookrightarrow \mathbb{C}^4, \rho \leftrightarrow (a_1, \ldots, d_2)$. To see what subset this is, we have to evaluate R on $\rho(x_i)$, so we get four polynomial equations with coefficients in $\mathbb{Z}$.

If we consider only irreducible representations , then we can conjugate to get rid of $\mathrm{SL}_2(\mathbb{C})$ action.

$$\rho(x_1) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix}, \rho(x_2) = \begin{pmatrix} \mu & 0 \\ r & \mu^{-1} \end{pmatrix}$$

Note that the relation in Γ implies that x_1 and x_2 are Γ-conjugate, so $\mu = \lambda^{\pm 1}$. Therefore $\rho(x_1)$ and $\rho(x_2)$ are $\rho(\Gamma)$-conjugate, and thus $\chi_\rho(x_1) = \chi_\rho(x_2) \ \forall \ \rho$.

Assume, since it's Tuesday, that $\mu = \lambda$ and evaluate

$$\rho([x_1^{-1}, x_2] x_1 [x_1^{-1}, x_2]^{-1}) - \rho(x_2) = 0$$

We get $w_2 = -1 + r + 3\lambda^2 - 3r\lambda^2 + r^2\lambda^2 - \lambda^4 + r\lambda^4$. Since we're only concerned with irreducible representations , assume $r \neq 0$. Given irreducible representation

, we obtain a point on $\mathbb{C} = \{\rho(x,r) = 0\} \subset \mathbb{C}^2$. Conversely, given a point in $\mathbb{C}$, we get an irreducible representation .

Set $\lambda = 1$, so $w_2 = -1 + r + 3 - 3r + r^2 - 1 + r = r^2 - r + 1$. Notice that $w_2 = 0 \iff r = \frac{1 \pm \sqrt{-3}}{2}$.

This is a discrete representation (since r is a quadratic imaginary integer). Also, it is faithful, but to see that we need topology.

By Lemma 1.5, any character χ_ρ is determined by $\chi_\rho(x_1) = \chi_\rho(x_2) = \lambda + \lambda^{-1} = z$. Then $R = \chi_\rho(x_1 x_2) = \lambda^2 + \lambda^{-2} + r = z^2 - 2 + r$. So, to get an equation for characters, consider $\rho(x,r) = 0$:

$$\begin{aligned}
\lambda^4(r-1) + \lambda^2(r^2 - 3r + 3) + (r-1) &= 0 \\
\lambda^2(r-1) + (r^2 - 3r + 3) + \lambda^{-2}(r-1) &= 0 \\
(r-1)(\lambda^2 + \lambda^{-2}) + (r^2 - 3r + 3) = 0 &
\end{aligned}$$

Rewriting in terms of z, R, we have $q(z,R) = z^2(2-R) + (R^2 - R - 1)$, so ρ is an irreducible representation implies that $\chi_\rho \leftrightarrow \mathcal{X} = \{q(z,R) = 0\} \subset \mathbb{C}^2$

$$\begin{aligned}
(z(-2+R))^2 &= (R-2)(R^2 - R - 1) \\
y^2 &= (R-2)(R^2 - R - 1) \subset \mathbb{C}^2
\end{aligned}$$

is just a torus in $\mathbb{C}^2$.

note: For any $\gamma \in \Gamma, \chi_\rho(\gamma) \in \mathbb{C}[z,R]$, i.e. given such a γ, this determines $I_\gamma : \mathcal{X} \to \mathbb{C}, \chi_\rho \mapsto \mathrm{Tr}\rho(\gamma)$. Typically, this will define a nonconstant function on $\mathcal{X}$. (Recall: for a Riemann surface, the number of zeros equals the number of poles for a nonconstant function.)

If F is a free group of rank 2, then $X(F) = \mathbb{C}^3$. The reducible representations correspond to the plane given by $r = 0$.

1.4. **Jan. 29: Algebraic geometry.** Setup: let $\Gamma =< G|\mathcal{R} >=< G'|\mathcal{R}' >$ with $G = \{\gamma_1, \ldots, \gamma_n\}, G' = \{\delta_1, \ldots, \delta_m\}$ and $\delta_i = w_i(\gamma_1, \ldots, \gamma_n), \gamma_j = v_j(\delta_1, \ldots, \delta_m)$. Recall diagram 1.3, and note that both $f \circ ({}^{-1}g)$ and $g \circ ({}^{-1}f)$ are polynomial maps, and $\rho(\delta_i) = W_i(\gamma_1, \ldots, \gamma_n)$ is a matrix with polynomial entries.

Definition 4. *Let $V \subset \mathbb{C}^n, W \subset \mathbb{C}^m$ be algebraic sets. Then a map $\varphi : V \to W$ is called a* ***polynomial map*** *if $\varphi(x_1, \ldots, x_n) = (f_1(x), \ldots, f_m(x))$ where each f_i is a polynomial map (i.e. each is a polynomial in $\mathbb{C}^n$ restricted to V).*

V and W are isomorphic if there exists $\alpha : V \to W, \beta : W \to V$ with α and β polynomial maps such that $\beta \circ \alpha \equiv \mathrm{id}|V, \alpha \circ \beta \equiv \mathrm{id}|W$. We can forget about generating sets and talk about $R(\Gamma)$ = the representation variety.
exercise: $X = \{(x, y) : x^3 - y = 0\}, X \cong \mathbb{C}$. Write down polynomial maps $X \to \mathbb{C}, \mathbb{C} \to X$ that compose appropriately. For example, $(x, y) \mapsto x$ and $t \mapsto (t, t^3)$.
Remarks.

(1) If $\varphi : \Gamma_1 \twoheadrightarrow \Gamma_2$ is a homomorphism and $\rho \in R(\Gamma_2)$ then $\rho \circ \varphi : \Gamma_1 \to \mathrm{SL}_2(\mathbb{C})$ defines a point in $R(\Gamma_1)$. Therefore it induces $\Phi : R(\Gamma_2) \to R(\Gamma_1), \rho \mapsto \rho \circ \varphi$.

examples:

(1) Let B denote the Borromean rings, so $\pi_1(S^3 \setminus B) \to F_2$, and there are lots of representations $F_2 \to \mathrm{SL}_2(\mathbb{C})$.
(2) Let K be a knot in $\#_r(S^2 x S^1) = X, r \geq 2$. Then $\pi_1(X \setminus K) \twoheadrightarrow \pi_1(X) \cong F_r$.
(3) If $M \to N$ is degree 1, then $\rho : \pi_1(N) \to \mathrm{SL}_2(\mathbb{C})$.

Remarks.

(1) If $\Gamma_2 \subset \Gamma_1$, and $\rho \subset \Gamma_1 \to \mathrm{SL}_2(\mathbb{C})$, then $\rho|_{\Gamma_2} : \Gamma_2 \to \mathrm{SL}_2(\mathbb{C})$. This defines a map $R(\Gamma_1) \to R(\Gamma_2)$.

Question: When are algebraic varieties the same?

Let V be an algebraic set in $\mathbb{C}^n$. Then $I(V)$ is an ideal in $\mathbb{C}[x_1, \ldots, x_n]$.

Definition 5. *The* ***coordinate ring*** *for V is*

$$\mathbb{C}[V] \;=\; \mathbb{C}[x_1, \ldots, x_n]/I(V)$$

note: $\mathbb{C}[V] \cong \{\varphi : V \to \mathbb{C} | \varphi \mathrm{apolynomial}\}$. Also, if $\phi | \mathbb{C}[X_1, \ldots, X_n] \to S$, then $\ker \phi = I(V)$.

The images of the coordinate functions in $\mathbb{C}[x_1, \ldots, x_n]$ generate $\mathbb{C}[V]$.
Remarks.

(1) V is a variety $\Leftrightarrow$ $I(V)$ is a prime $\Leftrightarrow$ $\mathbb{C}[V]$ is an integral domain.

If $\varphi : V \to W$ is a polynomial map between algebraic sets, then φ induces

$$\varphi_* : \mathbb{C}[W] \to \mathbb{C}[V]$$

by pullback.

$$\varphi_*(\beta) \;=\; \beta \circ \varphi : V \to \mathbb{C}$$

Furthermore V and W are isomorphic implies that $\mathbb{C}[V] \cong \mathbb{C}[W]$. (The converse is true if the isomorphism is the identity on constant polynomials.)

If V is irreducible then $\mathbb{C}[V]$ is an integral domain.

Definition 6. *The field of **rational functions** (or function field) on V is the quotient field for $\mathbb{C}[V]$.*

$$\mathbb{C}(V) \ = \ \left\{ \frac{f}{g} \ : \ f, g \in \mathbb{C}[V] \right\}$$

Note that $\frac{f}{g}$ takes a well-defined value in $\mathbb{C}$ for every point in V where g is non-zero.

The Zariski topology on V is generated by the algebraic sets as closed sets. Notice that if U and U' are non-empty open sets in a variety V, then $U \cap U' \neq \emptyset$. Otherwise V can be expressed as the non-trivial union of $V - U$ and $V - U'$, hence V is not irreducible.

By considering the open sets U and $V - \overline{U}$, we can establish the following claim. **claim:** If U is a non-empty, Zariski open subset of a variety V, then U is dense in V.

Pf. of claim: Consider $C = \overline{U} \subset V$, so $V \setminus C$ is open and $U \cap (V \setminus C) = \emptyset$. By the above note, and since $U \neq \emptyset$, $V \setminus C = \emptyset$, and therefore $C = V$. ⋄

Now we are prepared to make sense of the following definition.

Definition 7. *A **rational mapping** between algebraic varieties V and W is a map $\varphi : V \to W$ given as follows*

$$\varphi(x_1, \ldots, x_n) \ = \ \left(\frac{f_1(x_1, \ldots, x_n)}{g_1(x_1, \ldots, x_n)}, \ldots, \frac{f_m(x_1, \ldots, x_n)}{g_m(x_1, \ldots, x_n)} \right)$$

Where $f, g \in \mathbb{C}[x_1, \ldots, x_n]$.

The ambiguity comes in because φ is not defined where the g_i are zero. However, φ is defined on a dense subset of V by the above discussion.

Definition 8. *V and W are **birational** if there exist rational maps $\varphi : V \to W$ and $\psi : W \to V$ where their compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identities on their domains of definition.*

example:

$X = \{(R, z) : z^2(2-R) + (R^2 - R - 1) = 0\} Y = \{(x, y) : y^2 - (x-2)(x^2 - x - 1) = 0\}$

So X and Y are birational since we can take $x = R, y = (R-2)z$, which holds iff $z = \frac{y}{x-2}$. Using this gives $X \to Y, (R, z) \mapsto (R, z(R-2))$ and $Y \to X, (x, y) \mapsto (x, y/(x-2))$. Like the case of the coordinate ring, we get that V, W are birational iff $\mathbb{C}(V) \cong \mathbb{C}(W)$.

It may be useful to consider the similarity between the $\mathbb{Q}$ as the field of fractions over $\mathbb{Z}$ with the p-adic valuation vs. the function field $\mathbb{C}(V)$ over $\mathbb{C}[V]$ with the valuation corresponding to the orders of poles and zeros. In order to make sense of this, we need to say something about projective varieties.

1.5. **Feb. 3: Dimension of a variety and $X(\Gamma)$ is a subset of $\mathbb{C}^N$.** Let V be an algebraic set and associate to V the coordinate ring $\mathbb{C}[V]$. If V is also irreducible, then we can form the function field $\mathbb{C}(V)$.

Definition 9. *The* ***dimension*** *of a variety V is the transcendence degree of the extension $\mathbb{C}(V)/\mathbb{C}$. It is denoted $\dim_{\mathbb{C}} V$ or $\dim V$.*

NOTES ON TRANSCENDENCE:

Let K/F be an extension of fields.

Definition 10. *$\alpha \in K$ is* ***transcendental*** *over F if α is not algebraic over F; that is, α is transcendental if α does not satisfy a non-zero polynomial in $F[x]$.*

Let $S = \{\alpha_1, \ldots, \alpha_n\} \subset K$.

Definition 11. *S is* ***algebraically independent*** *if there does not exist a non-zero polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that $f(S) = 0$. If S is infinite, we say that S is algebraically independent if all finite subsets of S are algebraically independent.*

Definition 12. *A* ***transcendence basis*** *is a maximal algebraically independent set for K/F.*

Theorem 1.6. *K/F has a transcendence basis. Any two transcendence bases have the same cardinality.*

This allows us to make the following definition.

Definition 13. *The* ***transcendence degree*** *of K/F is the cardinality of any transcendence basis.*

examples:

(1) $V = \mathbb{C}^n$, $\mathbb{C}(x_1, \dots, x_n)$ = function field, so $\dim V = n$.
(2) $\dim V = 1 \iff V$ is a **curve**.
(3) Let $f(x, y) = y^2 - x^3 + x$ and $V = V(\langle f \rangle)$.

Since f is irreducible f is prime and therefore V is a variety.

Claim: $\dim V = 1$. (There exists a transcendence basis with one element.)

$$\pi : \mathbb{C}[x, y] \to \mathbb{C}[V]$$

Let $u = \pi(x)$ and $v = \pi(y)$. Then u and v generate $\mathbb{C}[V]$ over $\mathbb{C}$.

$$\mathbb{C}[V] = \mathbb{C}[u, v] \qquad \text{and} \qquad \mathbb{C}(V) = \mathbb{C}(u, v)$$

We have $v^2 = u^3 - u$ in $\mathbb{C}[u, v]$ so v is algebraic over $\mathbb{C}(u)$.

Claim: $\{u\}$ is a transcendence basis for $\mathbb{C}(u, v)$. So, we need to show that u is not algebraic over $\mathbb{C}$.

Note that $\mathbb{C}[u, v]$ is not a field since $\langle f \rangle$ is a proper ideal in $\langle x, y \rangle$ and hence, is not maximal. On the other hand, if u is algebraic then $\mathbb{C}[u, v]$ is an algebraic extension of $\mathbb{C}$. Then if $t \in \mathbb{C}[u, v]$ we know that there exist coefficients $a_i \in \mathbb{C}$ so that

$$\left(\frac{1}{t}\right)^n + a_{n-1}\left(\frac{1}{t}\right)^{n-1} + \cdots + a_0 \;=\; 0$$

Then

$$\frac{1}{t} \;=\; -(a_{n-1} + \cdots + a_0 t^{n-1}) \;\in\; \mathbb{C}[u, v]$$

We have established the contradiction that $\mathbb{C}[u, v]$ is a field.

The point is, in the situation of the last example, you can see how to write down the function field for the curve.

Let $\Gamma = \langle \gamma_1, \dots, \gamma_m \,|\, r_1, \dots, r_n \rangle$ and R_0 an irreducible component of $R(\Gamma)$.

We have a polynomial map between varieties

$$f : \Big(\mathrm{SL}_2(\mathbb{C})\Big)^m \to \Big(\mathrm{SL}_2(\mathbb{C})\Big)^n$$

by

$$(g_1, \dots, g_m) \;\mapsto\; \big(r_1(g_1, \dots, g_m), \dots, r_n(g_1, \dots, g_m)\big)$$

and

$$R(\Gamma) \;=\; f^{-1}(1, \dots, 1)$$

Theorem 1.7. *If V and W are varieties with dimensions m and n respectively. If $f : V \to W$ is a polynomial map and $w \in W$, then any irreducible component of $f^{-1}(w)$ has dimension bigger than or equal to $m - n$. (Mumford)*

Therefore, if we have a presentation for Γ as above, then $\dim R(\Gamma) \geq 3m - 3n$.

Remark. $\mathrm{SL}_2(\mathbb{C}) = \{(a, b, c, d) \in \mathbb{C}^4 : ad - bd - 1 = 0\}$ is irreducible. Similarly, $\mathrm{SL}_2(\mathbb{C})^m$ is also irreducible as an algebraic set.

If $\gamma \in \Gamma$ and $\rho \in R(\Gamma)$ then

$$\rho(\gamma) \;=\; \begin{pmatrix} a_\gamma(\rho) & b_\gamma(\rho) \\ c_\gamma(\rho) & d_\gamma(\rho) \end{pmatrix}$$

and we view $a_\gamma, b_\gamma, c_\gamma$, and d_γ as elements of $\mathbb{C}\big[R(\Gamma)\big]$. This leads to [what Culler-Shalen call] the tautological representation

$$\mathcal{P} : \Gamma \to \mathrm{SL}_2(F)$$

where $F = \mathbb{C}(R_0)$ and

$$\rho(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$$

Since ρ is a homomorphism, it follows that $\mathcal{P}$ is a homomorphism. Therefore $\mathcal{P} \subset \mathrm{SL}_2(\mathbb{C}(R_0))$ and this acts on $V = (\mathbb{C}(R_0))^2$.

Lemma 1.8. *If $\rho \in R_0$ is an irrep. then $\mathcal{P}$ is absolutely irreducible. (Irreducible over an algebraic closure.)*

Proof. If reducible, then the proof of Lemma 1.1 implies that $\mathrm{Tr}P(c) = 2\ \forall\ c \in [\Gamma, \Gamma]$. Therefore $\chi_\rho(c) = 2\ \forall\ c \in [\Gamma, \Gamma]$, so ρ is reducible, a contradiction. ☐

Theorem 1.9. *$X(\Gamma)$ is an algebraic set defined over $\mathbb{Q}$.*

Proof. Omitted. ☐

Let $t : R(\Gamma) \to X(\Gamma)$ be the map $\rho \mapsto \chi_\rho$. Define the function $I_\gamma : X(\Gamma) \to \mathbb{C}$ for every $\gamma \in \Gamma$ by $I_\gamma(\chi) = \chi(\gamma)$. Also define $\tau_\gamma = I_\gamma \circ t : R(\Gamma) \to \mathbb{C}$.

τ_γ is a polynomial in the ambient coordinates for $R(\Gamma)$ so $\tau_\gamma \in \mathbb{C}\big[R(\Gamma)\big]$. Let $T(\Gamma)$ be the subring generated by $S = \{\tau_\gamma\}_{\gamma\in\Gamma}$.

Theorem 1.10. *$T(\Gamma)$ is generated by a finite subset of S. In fact it is generated by the set*

$$\{\, \tau_\gamma \,|\, \gamma = \gamma_{i_1} \cdots \gamma_{i_k} \text{ where } 1 \le k \le n \text{ and } 1 \le i_1 < \cdots < i_k \le n \,\}$$

Proof. From Day 7, 2/10/04. We'll show $T(\Gamma) = T_0$ where T_0 is generated by elements of the form $\tau_{\gamma_{i_1}}\tau_{\gamma_{i_2}} \ldots \tau_{\gamma_{i_r}}, i_1, \ldots, i_r$ distinct integers in $\{1, \ldots, n\}$. In addition, we'll also show that $\forall\ \gamma \in \Gamma, \tau_\gamma \in T_0$.

For all $x, y \in \mathrm{SL}_2(\mathbb{C}), \mathrm{Tr}x\mathrm{Tr}y = \mathrm{Tr}xy + \mathrm{Tr}xy^{-1}$ (by computation). Therefore:

$$g, h \in \Gamma, \tau_g\tau_h = \tau_{gh} + \tau_{gh^{-1}}. \tag{1}$$

Step #1 Show that $\tau\gamma \in T_0$ if $\gamma = \gamma_{i_1}^{m_1} \ldots \gamma_{i_r}^{m_r}$ where $i_1, \ldots, i_r$ are distinct.
We'll induct on $\nu(\gamma) = \sum_{j=1}^{r} K_j$ where

$$K_j = \begin{cases} -m_j & \text{if } m_j \le 0 \\ m_j - 1 & \text{if } m_j > 0 \end{cases}$$

Case $\nu = 0$: In this case, all m_j are 1 or 0, so done by constuction.

Case $\nu > 0$: Suppose $m_r = 1$ and write $\gamma = \gamma_{i_1}^{m_1} \ldots \gamma_{i_r}^{m_r=1}$. Then there exists some $m_j \ne 1$, so choose the largest s so that $m_s \ne 1$. Let

$$\begin{aligned} \gamma' &= (\gamma_{i_{s+1}}^{m_{s+1}} \ldots \gamma {i_r}^{m_r})(\gamma)(\gamma_{i_{s+1}}^{m_{s+1}} \ldots \gamma {i_r}^{m_r})^{-1} \\ &= \gamma_{i_{s+1}}^{m_{s+1}} \ldots \gamma_{i_r}^{m_r}\gamma_{i_1}^{m_1} \ldots \gamma_{i_s}^{m_s}, m_s \ne 1 \end{aligned}$$

But $\tau_\gamma = \tau_{\gamma'}, \nu(\gamma) = \nu(\gamma')$, so we can assume $m_r \ne 1$. Now if $m_r > 1$, set $g = \gamma\gamma_{i_r}^{-1}$ and $h = \gamma_{i_r}$, so by 1 we have

$$\begin{aligned} \tau_{\gamma\gamma_{i_r}^{-1}}\tau_{\gamma_{i_r}} &= \tau_\gamma + \tau_{\gamma\gamma_{i_r}^{-2}} \\ \tau_\gamma &= \tau_{\gamma\gamma_{i_r}^{-1}}\tau_{\gamma_{i_r}} - \tau_{\gamma\gamma_{i_r}^{-2}} \end{aligned}$$

In order to apply the inductive hypothesis, we need to show that $\nu(\gamma\gamma_{i_r}^{-1}), \nu(\gamma\gamma_{i_r}^{-2}) < \nu(\gamma)$. But

$$\nu(\gamma\gamma_{i_r}^{-1} = \nu(\gamma_{i_1}^{m_1}\dots\gamma_{i_r}^{m_r-1}) = \nu(\gamma) - K(\gamma_{i_r}^{m_r}) + K(\gamma_{i_r}^{m_r-1})$$

Similarly, $\nu(\gamma\gamma_{i_r}^{-2}) = \nu(\gamma) - 2$ if $m_r \neq 2$ or $\nu(\gamma) - 1$ if $m_r = 2$. (The $m_r < 0$ case is similar.)

Let $g = \gamma\gamma_{i_r}, h = \gamma_{i_r}^{-1}$ and use 1 to write $\tau_{\gamma\gamma_{i_r}}\tau_{\gamma_{i_r}^{-1}} = \tau_\gamma + \tau_{\gamma\gamma_{i_r}^2}$. We must show that $\nu)\gamma\gamma_{i_r}), \nu(\gamma\gamma_{i_r^2}) < \nu(\gamma)$.

Finally, let $\gamma = \gamma_{i_1}^{m_1}\dots\gamma_{i_r}^{m_r}$ where i_r are not necessarily unique, and induct on r:

Case r=0. Done.

Case r¿0. We can assume that the i_j's are not unique. Notice taht we may replace γ by a conjugate such that $\gamma_{i_r} = \gamma_{i_s}$ for $s < r$, so ν is unchanged. Setting

$$V = \gamma_{i_1}^{m_1}\dots\gamma_{i_s}^{m_s}, W = \gamma_{i_{s+1}}^{m_{s+1}}\dots\gamma_{i_r}^{m_r}$$

so $\gamma = VW$, and we have the following:

$$\begin{aligned} \gamma = VW, \quad VW^{-1} &= \gamma_{i_1}^{m_1}\dots\gamma_{i_s}^{m_s-m_r}\dots\gamma_{i_{s+1}}^{m_{s+1}} \\ \tau_V\tau_W &= \tau_{VW} + \tau_{VW^{-1}} \\ \therefore \tau_\gamma = \tau_{VW} &= \tau_V\tau_W - \tau VW^{-1} \in T_0 \end{aligned}$$

□

Let $\{\delta_1, \dots, \delta_N\} \subset \Gamma$ be such that $\{\tau_{\delta_i}\}$ generate $T(\Gamma)$.

$$R(\Gamma) \xrightarrow{\tau} \mathbb{C}^N, \rho \mapsto (\tau_{\delta_1}(\rho), \dots, \tau_{\delta_N}(\rho))$$

$R(\Gamma) \searrow X(\Gamma)$, $X(\Gamma) \nearrow \mathbb{C}^N$: bijection, $V = \{(\tau_{\delta_1}(\rho), \dots, \tau_{\delta_N}(\rho))\}$

If $\chi_\rho(\gamma) = \chi_{\rho'}(\gamma)\ \forall\ \gamma \in \Gamma$, then this defines the same point in V. Conversely, if $\tau(\rho) = \tau(\rho')$, then Theorem 1.10 yields $\chi_\rho(\gamma) = \chi_{\rho'}(\gamma)\ \forall\ \gamma \in \Gamma$.

Therefore we can identify $X(\Gamma) \equiv V \subset \mathbb{C}^N$. How C-S approach this is as follows:

Let RED $\subset R(\Gamma)$ = set of reducible representations. Let IR $\subset R(\Gamma)$ = collection of components containing an irreducible representation. Decompose $X(\Gamma) = t(\text{RED}) \cup t(\text{IR})$, where $t(\rho) = \chi_\rho$. Culler-Shalen do hard work to show that $t(\text{IR})$ is an algebraic set.

claim: $t(\text{RED})$ is an algebraic set.

Pf. of claim:

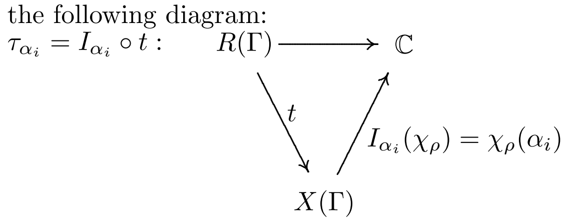
$$\begin{aligned} \chi_\rho \in t(\text{RED}) \quad &\text{iff} \quad \chi_\rho(c) = 2\ \forall\ c \in [\Gamma,\Gamma] \\ &\text{iff} \quad I_c(\chi_\rho) = 2\ \forall\ c \in [\Gamma,\Gamma] \\ &\text{iff} \quad \chi_\rho \in \{I_c(\chi_\rho) - 2 = 0\ \forall\ c \in [\Gamma,\Gamma] \end{aligned}$$

◇

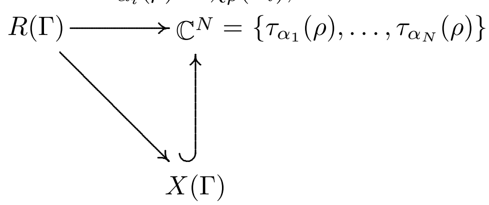
1.6. **Feb. 5: $X(\Gamma)$ is an algebraic set.** Let $\{\gamma_i\}_{i=1}^n$ be a generating set for Γ and F_n a free group with free basis $\{y_i\}_{i=1}^n$. Let $\pi : F_n \to \Gamma$ be the epimorphism where $\pi(\gamma_i) = y_i$ and let $y_0 = 1 \in F_n$. Take $W = \{w_j\}_{j \in J}$ to be a set of defining relations which correspond to $\{\gamma_i\}_{i=1}^n$. Then the normal closure of W in F_n is

$$\langle\langle W \rangle\rangle = \{y w_j y^{-1} \,|\, y \in F_n \text{ and } j \in J\}$$

From Theorem 1.10, there exists a collection $\alpha_1, \dots, \alpha_N \in \Gamma$ such that we get the following diagram:



...so $t_{\alpha_i}(\rho) = \chi_\rho(\alpha_i)$, and we have:



Any representation

$$\rho' : \Gamma \to \mathrm{SL}_2(\mathbb{C})$$

can be thought of as a representation

$$\rho : F_n \to \mathrm{SL}_2(\mathbb{C})$$

which factors through $\Gamma = F_n/\langle\langle W \rangle\rangle$ by π. Note that such a representation ρ factors through Γ if and only if

$$\rho(w_j y_i) \;=\; \rho(y_i) \qquad \forall\;\; j \in J \text{ and } \;\forall\;\; i = 0, \dots, n$$

Let p_{ij} be the polynomial with $\mathbb{Z}$ coefficients in the ambient coordinates $\{I_{\delta_i}\}_{i=1}^N$ given by

$$p_{ij}(\chi) \;=\; \chi(w_j y_i) - \chi(y_i)$$

So, if ρ factors through Γ, then $p_{ij}(\chi_\rho) = 0$ for every i and j.

Let $\mathfrak{X} \;=\; \{\chi \in X(F_n) \,|\, p_{ij}(\chi) = 0 \;\;\forall\;\; j \in Jand \;\forall\;\; i = 0, \dots, n\}$, and assume that $X(F_n)$ is an algebraic set.

Theorem 1.11. $X(\Gamma) \;=\; \mathfrak{X}$

Remarks.

(1) $X(\Gamma) \subset \mathfrak{X}$ by above discussion.
(2) notice that $P_{i0} : \chi_\rho(w_i) - 1 = 0$ but this does not imply that $\rho(w_i) = I$.

Proof. Let $\chi = \chi_\rho \in \mathfrak{X}$. We need to show that $\chi \in X(\Gamma)$.

$\rho : F_n \to \mathrm{SL}_2(\mathbb{C})$, let $\rho(y_j) = A_j, j = 1, \dots, n, \rho(y_0) = \mathrm{Id}$. By definition of $\mathfrak{X}$, we have:

$$\mathrm{Tr} w_i(A_1, \dots, A_n) A_j - 2 = 0 \;\forall\; i \in J, j = 0, \dots, n$$

In particular, this is true for $j = 0$, so

$$\mathrm{Tr} w_i(A_1, \dots, A_n) - 2 = 0 \ \forall \ i \in J$$

We may assume that $w_i(A_1, \dots, A_n) = \mathrm{Id}$ or $w_1(A_1, \dots, A_n) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ by conjugating ρ. We still have $\mathrm{Tr} w_1(A_1, \dots, A_n)A_j - \mathrm{Tr} A_j = 0, j = 1, \dots, n$. This is true if and only if

$$\mathrm{Tr}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} - \mathrm{Tr}\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} = 0, j = 1, \dots, n$$

which in turn is equivalent to $c_j = 0 \ \forall \ j = 1, \dots, n$. Therefore, each A_j is upper triangular, so ρ is a reducible representation.

Now replace ρ by a diagonal $\rho_1 : F_n \to \mathrm{SL}_2(\mathbb{C})$ and $\chi_{\rho_1} = \chi_\rho \in \mathfrak{X}$, and write $\rho_1(A_j) = \begin{pmatrix} a_j & 0 \\ 0 & d_j \end{pmatrix}$. Since $\chi_{rho_1} \in \mathfrak{X}$, we know that for each i, we have

$$\mathrm{Tr} w_i(\begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n & 0 \\ 0 & d_n \end{pmatrix}) - 2 = 0$$

But $w_i(\begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n & 0 \\ 0 & d_n \end{pmatrix})$ is diagonal, so $\rho_1(w_i) = 1$. Therefore ρ_1 defines a homomorphism ρ' that factors through Γ and $\chi_{\rho_1} \in X(\Gamma)$. □

Certainly, $X(\Gamma) \subset \mathfrak{X}$ from the above discussion. The issue is to show that if $\chi \in \mathfrak{X}$ then $\chi \in X(\Gamma)$. It turns out that if $\chi \in \mathfrak{X}$ corresponds to a representation ρ where $\rho(w_j)$ is a non-trivial parabolic for some i, then ρ is a reducible representation which has the same character as a representation that factors through Γ.

> QUESTION: Let X be any variety (defined over $\mathbb{Q}$). Is there a finitely generated, finitely presented group Γ for which X is an irreducible component of $X(\Gamma)$? Can Γ be taken to be a knot group?

Weak Culler-Shalen: Γ f.g. and suppose $X(\Gamma)$ contains an irreducible component of X with $\dim_{\mathbb{C}} X > 0$. Then Γ admits a splitting as a free product with amalgamation or an HNN extension.

SOME APPLICATIONS:

Using the following theorem from Mumford,

Theorem 1.12. *Assume that V and W are varieties and $\varphi : V \to W$ is a dominating polynomial map ($\overline{\varphi(V)} = W$). Then there exists a non-empty, Zariski open subset $Y \subset W$ such that for every $y \in Y$ the set $\varphi^{-1}(y) \neq \emptyset$.*

we can establish the following proposition

Proposition 1.13. *Let Γ be a finitely generated group and assume that $X(\Gamma)$ contains an irreducible component X_0 such that for every $\alpha \in \Gamma$ and every $\chi_\rho \in X_0$ $\chi_\rho(\alpha)$ is an algebraic number . Then* $\dim_{\mathbb{C}} X_0 = 0$.

Proof. Suppose $\dim_{\mathbb{C}} X_0 > 0$. As before, assume that $X(\Gamma)$ is parametrized by $\tau_{\delta_1}, \ldots, \tau_{\delta_N}$. Some I_{δ_j} is nonconstant on X_0; otherwise, since I_{δ_j} generate $\mathbb{C}[X(\Gamma)]$ (and, in particular, they generate $\mathbb{C}[X_0]$ as well), so $\mathbb{C}[X_0] = \mathbb{C} = \mathbb{C}(X_0)$, a contradiction.

claim: I_γ takes on a transcendental value for some $\gamma \in \Gamma$.

Pf. of claim: Using Theorem 1.12, $Y_0 \subset \mathbb{C}$ is Zariski open and not empty, so $Y_0 = \mathbb{C} \setminus V$ where V is at most a finite collection of points. Hence we can find a transcendental $t \in \mathbb{C}$ such that $I_\gamma(\chi) = t$, a contradiction. ◇

□

We can use Prop. 6.3 (?) to prove

Proposition 1.14. *Let Γ be a finitely generated, finitely presented group such that:*

1. *$X(\Gamma)$ contains an irreducible component X_0 of positive dimension;*
2. *there exists a faithful representation $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ so that $\chi_\rho \in X_0$;*
3. *and $\chi_\rho(\gamma) \neq 2$ for every $\gamma \in \Gamma - \{1\}$.*

Then there exists a faithful representation $\rho_1 : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ and an element $\gamma \in \Gamma$ where $\chi_{\rho_1} \in X_0$ and $\chi_{\rho_1}(\gamma)$ is transcendental.

Proof. Suppose ρ does not satisfy the conclusion. If $\varphi : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ is not injective, then there exists $1 \neq w \in \Gamma$ such that $\varphi(x) = I$, i.e. $\chi_\varphi(w) - 2 = 0$. Let X_w be the locus of $\{\chi_\rho(w) - 2 = 0\}$, which is an algebraic set, and let $X'_w = X_w \cap X_0$. This is a *subvariety* of X_0, with $X_0 \subset X(\Gamma) \subset \mathbb{C}^N$. This is a proper subvariety of X_0 (since $\chi_\rho(w) \neq 2$) by hypothesis 3. There is a countable number of words $w \in \Gamma$ (which is finitely generated) such that

$$Y = \cup_{w \in \Gamma} X'_w$$

We can now find a $\overline{\mathbb{Q}}$-generic point of X (i.e. a point lying on no proper subvariety over $\overline{\mathbb{Q}}$). Given such a point $\chi \in \mathbb{C}^N$, with $\chi = (\chi_1, \ldots, \chi_N)$, then at least one of the χ_i are transcendental. Hence $\chi = \chi_\rho$ is the required character and …is faithful. □

This leads to the following question.

QUESTION: Does there exist a closed, hyperbolic 3-manifold M with a faithful representation $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ where $\chi_\rho(\gamma)$ is transcendental for some $\gamma \in \Gamma$? (Note that discrete, faithful representations don't have transcendental traces.)

The issue is that the representation corresponding to the hyperbolic structure is isolated in $X\big(\pi_1(M)\big)$ since M is closed. To apply the previous proposition, we need $\dim_{\mathbb{C}} X_0 > 0$.

1.7. **Feb. 10: $T(\Gamma)$ is a finitely generated ring.** Recall Theorem 1.10: $T(\Gamma) \subset \mathbb{C}\big[R(\Gamma)\big]$ is the subring generated by $S = \{\tau_\gamma\}_{\gamma\in\Gamma}$ where $\tau_\gamma(\rho) = \mathrm{Tr}\big[\rho(\gamma)\big]$.

Theorem 1.15. *$T(\Gamma)$ is generated by a finite subset of S. In fact it is generated by the set*

$$\{\, \tau_\gamma \,|\, \gamma = \gamma_{i_1}\cdots\gamma_{i_k} \text{ where } 1 \le k \le n \text{ and } 1 \le i_1 < \cdots < i_k \le n \,\}$$

This can be proved by letting T_0 be the ring generated by the above functions, then showing that $\tau_\gamma \in T_0$ for every γ. First show that if

$$\gamma \;=\; \gamma_{i_1}^{m_1}\cdots\gamma_{i_r}^{m_r}$$

where the subscripts are all distinct then $\tau_\gamma \in T_0$. We can do this by induction on the complexity

$$\nu(\gamma) \;=\; \sum_{j=1}^{r} K_j$$

where

$$K_j \;=\; \begin{cases} -m_j & \text{if} \quad m_j \le 0 \\ m_j - 1 & \text{if} \quad m_j > 1 \end{cases}$$

Also use the trace relation

$$\mathrm{Tr}(AB) + \mathrm{Tr}(AB^{-1}) \;=\; \mathrm{Tr}(A)\cdot\mathrm{Tr}(B)$$

Now prove it for γ when the subscripts are not necessarily distinct by using the same trace relation and inducting on r.

1.8. **Feb. 12: The Tree for** $\mathrm{SL}_2(K)$**.** Let K be a field. Then a **valuation** $v : K \to \mathbb{Z}$ is a map such that $v(xy) = v(x) + v(y), xy \neq 0$ and $v(x+y) \geq \min(v(x), v(y)), x + y \neq 0$.

Let $v : K^\times \to \mathbb{Z}$ be a valuation on a field K. Then the valuation ring for v is

$$\mathcal{O} \;=\; \mathcal{O}_v \;=\; \{\alpha \in K \,|\, v(\alpha) \geq 0\}$$

$\mathcal{O}$ has a unique maximal ideal $\mathcal{M}_v$ which is generated by any element $\pi \in \mathcal{O}$ where $v(\pi) = 1$. (Note: π is called the **uniformizer**.) The residue field for v is the quotient $\mathfrak{k}_v = \mathcal{O}/\mathcal{M}_v$.

examples:

(1) $K = \mathbb{Q}$, $v_p : \mathbb{Q} \to \mathbb{Z}, x = \frac{a}{b} = \frac{a'}{b'}p^k, p \nmid a'$ or b'. Then $v_p(x) = k, \mathcal{O}_p = (p)$, and $k_p = \mathbb{Z}_p$.

(2) K = field of meromorphic functions on $\mathbb{C}$: $\mathcal{O}_{z_0} = \{f \in K : f(z_0) \neq \infty\}$

$$f(z) = g(z)(z - z_0)^m, v_{z_0}(f) = m, \text{ and } k = \mathcal{O}_{z_0}/\mathcal{O}_{z_0}(z - z_0) \cong \mathbb{C}$$

We can view the vector space $V = K^2$ as an $\mathcal{O}$-module.

Definition 14. *An* ***$\mathcal{O}$-lattice*** *in V is an $\mathcal{O}$-submodule which is finitely generated and spans V as a K vector space.*

If Λ is an $\mathcal{O}$-lattice in V then Λ is a rank 2 free $\mathcal{O}$-module. Also, if Λ_1 and Λ_2 are lattices with respective bases $\{e_i, f_i\}$, then there exists $A \in \mathrm{GL}_2(K)$ such that $A \cdot \{e_1, f_1\} = \{e_2, f_2\}$. Therefore, $A(\Lambda_1) = \Lambda_2$.

Lemma 1.16. *If Λ_1 and Λ_2 are $\mathcal{O}$-lattices in V and $A, B \in \mathrm{GL}_2(K)$ such that $A(\Lambda_1) = \Lambda_2$ and $B(\Lambda_1) = \Lambda_2$ then*

$$v\big(\det(A)\big) \;=\; v\big(\det(B)\big)$$

Proof. Show that $v\big(\det(B^{-1}A)\big) = 0$. □

If Λ_1 and Λ_2 are lattices and $A(\Lambda_1) = \Lambda_2$ then we make the definition

$$\delta(\Lambda_1, \Lambda_2) \;=\; v\big(\det(A)\big)$$

By the lemma, δ is well-defined. Also, δ has the following basic properties:

(a) $\delta(\Lambda, \Lambda) = 0$ because $v\big(\det(I)\big) = 0$.
(b) $\delta(\Lambda_1, \Lambda_3) = \delta(\Lambda_1, \Lambda_2) + \delta(\Lambda_2, \Lambda_3)$.
(c) If $\Lambda_1 \subset \Lambda_2$ then $\delta(\Lambda_2, \Lambda_1) \geq 0$.
(d) If $B \in \mathrm{GL}_2(K)$ then $\delta\big(B(\Lambda_1), B(\Lambda_2)\big) = \delta(\Lambda_1, \Lambda_2)$.

Definition 15. *Two lattices Λ_1 and Λ_2 are* ***homethety equivalent*** *(denoted $\Lambda_1 \sim \Lambda_2$) if there exists $\alpha \in K$ such that $\alpha \cdot \Lambda_1 = \Lambda_2$.*

Lemma 1.17. *If Λ_1 and Λ_2 are lattices then Λ_1 is equivalent to a lattice Λ_1' such that $\Lambda_1' \subset \Lambda_2$.*

Proof. Idea: Multiply by a large power of the uniformizer π.

Let $\Lambda_i = \{e_i, f_i\}$ be bases as $\mathcal{O}$-modules. Then there exist $\alpha, \beta \in k$ such that $e_2 = \alpha e_1 + \beta f_1$. Let $m_0 = -\min(v(\alpha), v(\beta))$. For any $m \geq m_0$

$$v(\pi^m \alpha) = v(\pi^m) + v(\alpha) \geq m_0 - m_0 = 0$$

□

Lemma 1.18. *If Λ_1 and Λ_2 are lattices then there exists a unique $\Lambda_1' \in [\Lambda_1]$ such that $\Lambda_1' \subset \Lambda_2$ and $\Lambda_2/\Lambda_1' \cong \mathcal{O}/\beta\mathcal{O}$ for some $\beta \in \mathcal{O} - \{0\}$. (Say that Λ_1' is **snuggly embedded** in Λ_2, denoted by $\Lambda_2 << \Lambda_1$.)*

Proof. Use the structure theorem for finitely generated modules over a PID. If $\Lambda_1 \subset \Lambda_2$ then there exists a basis $\{e, f\}$ for Λ_2 and non-zero $\alpha, \gamma \in \mathcal{O}$ such that $\{\alpha e, \gamma f\}$ generates Λ_1. WLOG $v(\alpha) \leq v(\gamma)$. Let $\Lambda_1' = \left(\frac{\gamma}{\alpha}\right)\Lambda_1$. Uniqueness is easier. □

If s_1 and s_2 are homothety classes of lattices, we define the distance between them as

$$d(s_1, s_2) \;=\; \delta(\Lambda_1, \Lambda_2)$$

where $\Lambda_i \in s_i$ and $\Lambda_2 << \Lambda_1$. We can think of this as taking $\Lambda_1 \in s_1$ and then $\Lambda_2 \in s_2$ so that $\Lambda_2 \subset \Lambda_1$ snugly. Then $\Lambda_1/\Lambda_2 \cong \mathcal{O}/\beta\mathcal{O}$ and $d(s_1, s_2) = v(\beta)$.

Lemma 1.19. *If $\Lambda_2 \subset \Lambda_1$ are lattices then*

(a) $d\big([\Lambda_1], [\Lambda_2]\big) \leq \delta(\Lambda_1, \Lambda_2)$
(b) $d\big([\Lambda_1], [\Lambda_2]\big) = \delta(\Lambda_1, \Lambda_2) \pmod 2$
(c) $d\big([\Lambda_1], [\Lambda_2]\big) = \delta(\Lambda_1, \Lambda_2)$ *if and only if Λ_2 is snugly embedded in Λ_1.*

Let $T^{(0)}$ be the set of homethety classes of lattices in V. Then the function $d : T^{(0)} \times T^{(0)} \to \mathbb{N}$ is well-defined.

Lemma 1.20. *$(T^{(0)}, d)$ is a metric space.*

Lemma 1.21. *For every $r, s, t \in T^{(0)}$ we have*

$$d(r, s) + d(s, t) = d(r, t) \pmod 2$$

Lemma 1.22. *If $r, t \in T^{(0)}$ then let $n = d(r, t)$. If $p + q = n$ (where p, q are positive integers) then there exists a unique $s \in T^{(0)}$ where $d(r, s) = p$ and $d(s, t) = q$.*

Let T be the graph formed by connecting every pair of points $s, t \in T^{(0)}$ with $d(s, t) = 1$ by an edge.

Theorem 1.23. *T is simply connected.*

Theorem 1.24. *$\mathrm{GL}_2(K)$ acts on T by isometries.*

If $\Lambda \sim \Lambda'$ are equivalent $\mathcal{O}$-lattices and $B \in \mathrm{GL}_2(K)$ then $B \cdot \Lambda \sim B \cdot \Lambda'$. Therefore, the action is well-defined on vertices. Also, we can use the fact that $\delta(\Lambda_1, \Lambda_2) = \delta(B \cdot \Lambda_1, B \cdot \Lambda_2)$ to show that B is an isometry.

The $\mathrm{GL}_2(K)$ action restricts to an action of $\mathrm{SL}_2(K)$ which has the following properties:

(1) If $B \in \mathrm{SL}_2(K)$ then $d\Big([\Lambda], B \cdot [\Lambda]\Big)$ is even.
(2) $\mathrm{Stab}_{[\Lambda]}\big(\mathrm{SL}_2(K)\big) = \mathrm{Stab}_{\Lambda}\big(\mathrm{SL}_2(K)\big)$

Note that neither of these are true for $\mathrm{GL}_2(K)$ since the proofs rely on the fact that $v\big(\det(B)\big) = v(1) = 0$.

By (1), we see that $\mathrm{SL}_2(K)$ acts without inversions. By (2), we see that every vertex stabilizer is conjugate to $\mathrm{SL}_2(\mathcal{O})$.

Theorem 1.25. *There is a bijection between the vertices in the link of any given vertex and the 1-dimensional subspaces of $\mathfrak{k}^2$, ie the elements of $\mathfrak{k}\mathrm{P}^1$ ($\leftrightarrow \mathfrak{k} \cup \{\infty\}$).*

For any edge e of T, the edge stabilizer $\Big(\mathrm{SL}_2(K)\Big)_e$ is conjugate to the group

$$\triangle = \left\{ \begin{pmatrix} a & b \\ c\pi & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}) \right\}$$

Also if G is the commutator subgroup $\left[\Big(\mathrm{SL}_2(K)\Big)_e, \Big(\mathrm{SL}_2(K)\Big)_e\right]$ and $A \in G$, then A is conjugate to a matrix of the form

$$\begin{pmatrix} a & b \\ c\pi & d \end{pmatrix}$$

where $a \equiv d \equiv 1 \pmod{\pi}$. Therefore

$$\mathrm{Tr}(A) = a + d \in (2 + 2\pi)$$

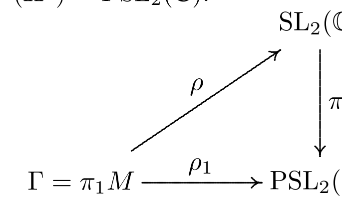
that is, $\mathrm{Tr}(A) \equiv 2 \pmod{\pi}$.

1.9. **Feb. 17: Dimension of the canonical component.**

examples:

(1) "interesting" examples arise from $\pi_1(M)$,
bolic 3-manifold. Set $\mathbb{H}^3 = \{(x, y, t) : t >$
equipped with the metric ρ given by ds^2
perbolic manifold is $\mathbb{H}^3/\Gamma$ where $\Gamma <$ Isor
torsion-free.

But $\text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}_2(\mathbb{C})$:



The irritation in this diagram is that ρ
der 2 but in $\text{SL}_2(\mathbb{C})$ only nontrivial ele
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Theorem 1.26 (Mostow Rigidity). *Let $M = \mathbb{H}^3/\Gamma$ and $N = \mathbb{H}^3/\Gamma'$ be isometric hyperbolic 3-manifolds. Then Γ and Γ' are conjugate in* $\text{Isom}(\mathbb{H}^3)$.

Mostow rigidity implies that the characters in $X(\Gamma)$ that correspond to discrete, faithful representations consist of a finite number of points.

Theorem 1.27 (Thurston). *Let M be a compact, orientable 3-manifold with non-empty boundary consisting of a disjoint union of n incompressible tori. Assume that* $\text{int}(M) = \mathbb{H}^3/\Gamma$. *Then if ρ is a discrete, faithful representation, there exists an irreducible component $X_0 \subset X(\Gamma)$ such that $\chi_\rho \in X_0$ and* $\dim_{\mathbb{C}} X_0 = n$.

We call X_0 the **canonical component**.

A good source for examples is $\Gamma = \pi_1(M)$ in two generators:

$$< a, b | R(a, b) = 1 > \text{ or } < a, b | R_i(a, b) = 1, i = 1, 2 >$$

examples:

(1) Consider a two-bridge knot or link; this has a presentation of the form $< a, b | waw^{-1} = b >$. Consider irreducible representations $\rho : \Gamma \to \text{SL}_2(\mathbb{C})$, and conjugate so

$$\rho(a) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix}, \rho(b) = \begin{pmatrix} \mu & 0 \\ r & \mu^{-1} \end{pmatrix}$$

note: If $\chi_\rho(a) = \chi_\rho(b)$, then $\mu = \lambda^{\pm 1}$. Now we have $\Gamma =< a, b | R(a, b) = 1 >, X(\Gamma) \subset \mathbb{C}^3$ since $X(\Gamma)$ is generated by $(\chi_\rho(a), \chi_\rho(b), \chi_\rho(ab))$. But for 2-bridge knots, $\chi_\rho(a) = \chi_\rho(b)$. Therefore $X_0 \subset X(\Gamma) \subset \mathbb{C}^2$ is generated by $(\chi_\rho(a), \chi_\rho(ab))$.

(Mathematica calculations...)

1.10. **Feb. 19: 3-Manifolds with high dimensional components in $X(\pi_1(M))$.** Let $F_n = \langle \alpha_1, \ldots, \alpha_n \rangle$ be the free group with rank $n \geq 2$. By assuming that ρ is an irreducible representation and normalizing $\rho(\alpha_1)$ and $\rho(\alpha_2)$ we have $3(n-2)+3 = 3n-3$ independent choices for matrix entries. Hence

$$\dim_{\mathbb{C}} X(F_n) \geq 3n-3$$

In particular, if a group surjects F_n where $n \geq 2$ then we are guaranteed components in the character variety with dimension at least $3n-3$.

Examples:

- Let $X_n = \#_{n \geq 2} \, (\mathrm{S}^1 \times \mathrm{S}^2)$. Then $\pi_1(X_n) \twoheadrightarrow F_n$.
- There exist closed hyperbolic 3-manifolds M with $\pi_1(M) \twoheadrightarrow F_n$.

  Exercise: There exists a knot K in X_n so that $X_n \backslash K$ is hyperbolic. Therefore, there exists hyperbolic 3-manifolds with high dimensional components.
- By the following lemma, there exist closed hyperbolic 3-manifolds M with high dimensional components.

  Lemma 1.28. *Let L be a link in S^3 with $|L| \geq 4$. Assume that by deleting one component of L we get the trivial link with $n-1$ components (i.e. L is Brunnian). Then $X(L)$ contains a component of dimension at least $3(n-1)-3 > n$.*

  Proof. The conditions of the lemma imply that $\pi_1(L)$ surjects F_{n-1}. □

  Downside: to construct hyperbolic knots K in S^3 with $X(K)$ containing a component of large dimension (i.e. at least 2), we cannot use this trick since $H_1(S^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z}$ implies that $\pi_1(S^3 \setminus K)$ does not surject F_n for $n \geq 2$.
- There exist knots with high dimensional components of $X(K)$.

  First let $K_1 = K = 4_1, \Gamma = \pi_1(s^3 \setminus K) =< a, b | waw^{-1} = b >$. We have seen that the canonical component X_0 is given by the locus of $\{P(z,R) = 0\}$ where $P(z,R) = z^2(2-R) + (R^2 - R - 1)$.

  Note that $\chi_\rho(a) = z$ can be chosen to be any $z \in \mathbb{C}$.

  Consider (irreducible) representations with

$$\rho(a) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \rho(b) = \begin{pmatrix} a_0 & 1 \\ c_0 & d_0 \end{pmatrix}$$

  where $c_0 \neq 0$ since ρ is irreducible.

  Let $T = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, t \in \mathbb{C} \setminus \{0\}$. Note that T commutes with $\rho(a)$.

  Let K' be a hyperbolic knot and $K = K' \# K'$. Write $\Gamma' = \pi_1(K')$ and $\Gamma = \pi_1(K)$. Then $\Gamma = \Gamma'_L *_{\mathbb{Z}} \Gamma'_R$ where the amalgamating subgroup is meridianal, say $\langle a \rangle$. Take a to be a generator in the presentation and conjugate ρ so that $\rho(a)$ is diagonal. Let T be any diagonal matrix in $\mathrm{SL}_2(\mathbb{C})$ so that $T\rho(a) = \rho(a)T$. Now given a representation ρ of Γ' we get a $\mathbb{C}^\times$ worth of representations of Γ by defining

$$\rho_L^t(\gamma) \;=\; \rho(\gamma) \qquad \text{for } \gamma \in \Gamma'_L$$

  and

$$\rho_R^t(\gamma) \;=\; T\rho(\gamma)T^{-1} \qquad \text{for } \gamma \in \Gamma'_R$$

Since ρ_L^t and ρ_R^t agree on $\langle a \rangle$ they paste together to give a representation

$$\rho_t : \Gamma \to \mathrm{SL}_2(\mathbb{C})$$

By setting $K_n = K' \# \ldots \# K_n$ and inducting on n we get $X(K_n)$ has a component of dimension at least n.

- There exist hyperbolic knots with high dimensional components in $X(K)$.

  Let A be the 5-braid $(\sigma_1\sigma_2^{-1})^2(\sigma_3\sigma_4^{-1})^2$.

  Then if K is the figure eight knot, $K_2 = \widehat{A}$ is $K\#K$. Let L_2 be the link $K_2 \cup B$ as shown.

  $\mathrm{S}^3 \backslash L_2$ is hyperbolic, as it is a 5-punctured disc bundle over S^1 with Psuedo-Anosov monodromy. For large enough p, $(p, 0)$-orbifold surgery on the B component of $\mathrm{S}^3 \backslash L_2$ will yield a hyperbolic orbifold. Hence there is a cyclic branched cover of $\mathrm{S}^3 \backslash K_2$ branched over B, call it M_p, which is a p-fold orbifold cover.

  M_p is hyperbolic as an orbifold cover of a hyperbolic orbifold. If p is chosen to be coprime to 5, then M_p is a knot complement. Now we have a degree p map $M_p \to \mathrm{S}^3 \backslash K_2$. Then $\pi_1(M_p)$ surjects an index p subgroup of $\pi_1(\mathrm{S}^3 \backslash K_2)$.

1.11. **Feb. 24: Finding group splittings algebraically.** Recall: a partial goal is to prove Weak Culler-Shalen: Γ f.g., f.p. group such that $X(\Gamma)$ contains an irreducible curve of characters of irreducible representations. Then Γ admits a splitting as a free product with amalgamation or as an HNN extension.

For $X_0(M_{\text{sis}})$ is parametrized by $(P, Q, R) = (P, 1 - P^2, (1 + P^2 - P^4)/P)$. Let $\Gamma = \pi_1(M_{\text{sis}})$, and take ρ so that

$$\rho(a) = \begin{pmatrix} 0 & 1 \\ -1 & 1/2 \end{pmatrix}, \rho(b) = \begin{pmatrix} 1/4 & 1/2 \\ -7/4 & 1/2 \end{pmatrix}$$

so $\rho(\Gamma) \subset \mathrm{SL}_2(\mathbb{Z}[1/2])$. We like this since we get an action on a tree T_2 of $\mathrm{SL}_2(\mathbb{Q})$. We have the rationals with the 2-adic valuation, $(\mathbb{Q}, v_2)$. Vertices are equivalence classes $[\Lambda]$ of $\mathbb{Z}_{(2)}$-lattices, where

$$\mathbb{Z}_{(2)} = \{\alpha \in \mathbb{Q} : v_2(\alpha) \geq 0\}$$

There is an edge between $[\Lambda]$ and $[\Lambda']$ if there exist representative lattices Λ, Λ' such that $\Lambda/\Lambda' \cong \mathbb{F}_2$. Note: vertices of T_2 have valence 3.

Consider the action of the subgroup $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{2}])$ on T_2. Since $\mathcal{O} = \{x \in \mathbb{Q} \,|\, \mathrm{ord}_2(x) \geq 0\}$, then $\mathcal{O} \cap \mathrm{SL}_2(\mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}$. Hence, every vertex stablizer is conjugate to $\mathrm{SL}_2(\mathbb{Z})$. Here we see that the action is non-trivial since otherwise the entire group would be conjugate into $\mathrm{SL}_2(\mathbb{Z})$.

Let v be the vertex stabilized by $\mathrm{SL}_2(Z)$ and $\{e, f\}$ a basis for a representative lattice. Then $\{e, 2f\}$ is a basis for a lattice representing a vertex v' where $d(v, v') = 1$. Let e be the edge connecting these two vertices. Then the edge stabilizer is

$$\Big(\mathrm{SL}_2(\mathbb{Z}[1/2])\Big)_e = \Gamma_0(2) = \left\{\begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\right\}$$

By Serre, this gives rise to a splitting

$$\mathrm{SL}_2(\mathbb{Z}[1/2]) = \mathrm{SL}_2(\mathbb{Z}) *_{\Gamma_0(2)} G(2).$$

We can use the same idea to get a splitting of the fundamental group of the sister to the figure eight,

$$\Gamma = \langle a, b \,|\, a^2ba^{-1}b^3 3a^{-1}b \rangle.$$

Let M be finite volume hyperbolic with one cusp and T be a peripheral torus. $M = \mathbb{H}^3/\Gamma$ and from the Dehn Surgery Theorem we know that $X_0 = X_0(\Gamma)$ is a curve.

Theorem 1.29 (Thurston)**.** *Let $\gamma \in \pi_1(T)$ (non-trivial). Then the function I_γ is nonconstant on X_0.*

Now using the above fact that I_γ is nonconstant, we can find a non-integral rational point x. (ie. $x \in \mathbb{Q} \backslash \mathbb{Z}$) so that I_γ^{-1} is a finite collection of points in X_0. Amongst these guys we can find χ_ρ such that $\rho(\Gamma) \subset \mathrm{SL}_2(K)$ where K is a finite extension of $\mathbb{Q}$ and $\rho(\gamma)$ has non-integral trace. As before, we get $\rho_* : \Gamma \to \mathrm{SL}_2(K_v)$ where v is a valuation associated to a prime divisor of the denominator of x.

The point is that we want to organize how we can see all splittings in a useful way.

Warm up # 10: tautological representation gives us $P : \Gamma \to \mathrm{SL}_2(\mathbb{C}(R_0))$, with $P(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$:

$$\begin{array}{ccc} R_0 & & R(\Gamma) \\ \Big\downarrow t & & \Big\downarrow t \\ X_0 & \xhookrightarrow{\text{incl.}} & X_\Gamma \end{array}$$

If we want to repeat above analysis of sister using $(\mathbb{Q}, v_2)$ with $(\mathbb{C}(R_0), v)$, v a valuation, we need to understand:

(i) valuations
(ii) what it means to be negative of valuations

1.12. Feb. 26: Valuations and Projective completions.

Definition 16. *If K is a field, then a* ***valuation*** $v : K^\times \to \mathbb{Z}$ *is an epimorphism between groups with the property*

$$v(x+y) \geq \min\big(v(x), v(y)\big).$$

v may be extended to all of K by $v(0) = \infty$.

> **examples:**
> 1. $v_p : \mathbb{Q}^* \to \mathbb{Z}$.
> 2. $v_p : \mathbb{C}(z)^* \to \mathbb{Z}$ using the fact that $F \in \mathbb{C}[z]$ implies that $F(z) = (z-p)^n F_0(z), n \geq 0, (z-p) \nmid F_0(z)$.

Now, $\mathbb{Z}_{(p)} = \{x \in \mathbb{Q} : v_p(x) \geq 0\}$ is a local ring with a unique maximal ideal $p\mathbb{Z}_{(p)} = \{x \in \mathbb{Q} : v_p(x) > 0\}$. Similarly, we have $\mathcal{O}_p$ and $p\mathcal{O}_p = M_p = \{F \in \mathbb{C}(z) : v_p(F) > 0\}$.

We can extend this idea to irreducible curves V with function field $\mathbb{C}(V)$. Let V be a (irreducible) curve and $p \in V$. We define the local ring at p as

$$\mathcal{O}_p \;=\; \left\{ \frac{f}{g} \mid f, g \in \mathbb{C}[V],\; g(p) \neq 0 \right\}.$$

$\mathcal{O}_p$ has a unique maximal ideal

$$\mathcal{M}_p \;=\; \left\{ \frac{f}{g} \mid f, g \in \mathbb{C}[V],\; g(p) \neq 0,\; f(p) = 0 \right\}.$$

Theorem 1.30 (Hartshorne)**.** *$\mathcal{O}_p$ is a DVR with valuation $v_p(F) = \mathrm{ord}_p(F)$ if and only if p is a smooth point of V.*

note: Say that a point is **smooth** iff $\mathrm{rk}\left[\frac{\partial f_i}{\partial x_j}(p)\right] = n - \dim(V)$; otherwise, it's **singular**.

> **examples:**
> 1. $V \subset \mathbb{C}^2$ irreducible and $\{f(x,y) = 0\} = V$ implies that $p \in V$ is singular iff $\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = 0$.
>
>   Fact: the set of singular points forms a proper subvariety of V.
> 2. $f(x,y) = x^2(2-y) + (y^2 - y - 1)$ (figure 8 again). Computing $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ shows that this has no singular points.

It may be useful to ponder the following analogy, especially if such pondering is done is a whistful manner:

cf.	$\mathbb{Q}$	$\mathbb{C}(V), V$ a smooth, irreducible curve
valuations:	v_p	$v_p, p \in V$
local rings:	$\mathbb{Z}_{(p)}$	$\mathcal{O}_{p,V}$
max'l ideals:	$p\mathbb{Z}_{(p)}$	M_p

Now if R_0 is a smooth curve in the representation variety $R(\Gamma)$ such that $t : R_0 \to X_0$. Then for every point p in R_0 we get a valuation v_p on $\mathbb{C}(R_0)$. The action of Γ (via the tautological representation) will always be trivial since $\mathrm{Tr}\big(\mathcal{P}(\gamma)\big)$ is a polynomial. This means that its valuation under v_p will never be negative (no poles) and every element stabilizes a vertex. It can be shown that if every element stabilizes a vertex, then there is some vertex that is fixed by the entire group. Hence the action given on this tree is trivial. We see here that we must pass to the projective model to get non-trivial actions.

Definition 17. *$V \subset \mathbb{CP}^n$ is a* ***projective algebraic set*** *if $V =$ the vanishing set of $I(V) \subset \mathbb{C}[X_0, \ldots, X_n]$ of homogeneous polynomials. V is a* ***projective algebraic variety*** *if $I(V)$ is prime.*

Let φ_i be the standard coordinate charts for $\mathbb{CP}^n$, $1 \leq i \leq n$. The hope is that $V \subset \mathbb{C}^n$ is an algebraic variety only if $\varphi_i(V) \subset \mathbb{CP}^n$ is a *projective* algebraic variety.

Definition 18. *A* ***projective completion*** *(or* ***closure****) of V an algebraic variety is a projective variety defined by the ideal*

$$I(V) \;=\; \{\, f^*(x_0, \ldots, x_n) \,|\, f \in I(V)\,\}\,.$$

example: Consider $f(x,y) = x^2(2-y) + (y^2 - y - 1), X_0 = \{f(x,y) = 0\}$, and let the map $X_0 \to \mathbb{CP}^2$ be given by $(x,y) \mapsto [x : y : 1]$. Then

$$\begin{aligned} f*(x,y,z) &= z^3 f\left(\frac{x}{z}, \frac{y}{z}\right) \\ &= 2x^2z - x^2y + y^2z - yz^2 - z^3 \end{aligned}$$

In general, f^* is f homogenized in the i^{th} coordinate.

Definition 19. *The points of $\overline{V} \backslash V$ are called* ***points at infinity*** *(or* ***ideal points****).*

example: X_0 (the figure 8): to get points at infinity, set $z = 0$ and we obtain $yx^2 = 0$, so $x = 0$ or $y = 0$. Therefore we get two points at infinity: $[0 : 1 : 0]$ and $[1 : 0 : 0]$.

Theorem 1.31. *If C is an affine irreducible curve, then there exists a smooth projective curve $\widetilde{C}$ such that $\mathbb{C}(C) \cong \mathbb{C}(\widetilde{C})$. $\widetilde{C}$ is unique up to birational equivalence.*

1.13. **March 2: The Splitting Theorem.** Recall: If $V \subset \mathbb{C}^n$ is an irreducible algebraic curve, then we can associate to V a projective curve $\overline{V}$ called the projective completion. This is achieved in several ways, via the maps $\varphi_i : \mathbb{C}^n \to \mathbb{CP}^n, \varphi(a_1, \ldots, a_n) = [a_1 : a_2 : \ldots : a_i : 1 : a_{i+1} : \ldots : a_n]$. Set $U_i = \varphi_i(\mathbb{C}^n)$, so $\cup_{i=0}^n U_i = \mathbb{CP}^n$.

We also have hyperplanes $H_i = \mathbb{CP}^n \setminus U_i = \{[x_0 : \ldots : 0 : \ldots : x_n]\}$ (where the 0 is in the $i^{mathrmth}$ coordinate). The variety $\overline{V}$ is defined by the homogeneous ideal $I(\overline{V}) = \{f^*(x_0, \ldots, x_n) : f \in I(V)\}$ where $f^*(x_0, \ldots, x_n) = x_i^d f(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i})$ where d is the degree of f. Points in $\overline{V} \setminus V$ are called points at infinity.

Definition 20. *A point $p \in \overline{V}$ is **smooth** if we can find U_i such that $p \in \overline{V} \cap U_i$ is smooth in the previous sense.*

example: Dehomogenize with respect to y (in last example), i.e. $F(x,z) = f^*(x,1,z) = 2x^2z - x^2 + z - z^2 - z^3$. Check $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial z}$ at the point $[0:1:0]$.

Key point: If V is an irreducible curve in $\mathbb{C}^n$, there is a **smooth model** $\widetilde{V}$ for V that is the desingularization of $\overline{V}$. This $\widetilde{V}$ is unique up to birational equivalence, and $\mathbb{C}(\widetilde{V}) \cong \mathbb{C}(V)$. $\widetilde{V}$ is a compact Riemann surface (here, compact means closed and without boundary), and the genus can be computed. If $\widetilde{V} \subset \mathbb{CP}^2$ is a smooth projective curve, then the genus is $\frac{(d-1)(d-2)}{2}$ where d is the degree of F and $\widetilde{V}$ is defined by $\{F(x,y,z) = 0\}$.

example: The genus of $\widetilde{X_0}$ of the figure-eight knot is 1, and we have maps $I_\gamma : X_0 \to \mathbb{C}$ and $I_\gamma : \widetilde{X_0} \to \mathbb{CP}^1$.

Theorem 1.32. *Let V be a smooth projective curve, $\varphi \in \mathbb{C}(V)$. Then:*

1. *φ has a finite number of zeros and a finite number of poles (i.e. points x where $v_x(\varphi) < 0$);*
2. *There is the same number of zeros and poles;*
3. *If φ has no poles, then φ is constant.*

example: $f(x,y) = x^2(2-y) + (y^2 - y - 1), \Gamma_8 =< a, b | waw^{-1} = b >$. Now $I_a(\chi_\rho) = \chi_\rho(a) = x$ has 2 zeros on X_0 and so 2 poles on $\widetilde{X_0}$. These poles occur at points at infinity.

Remark. If $V \subset \mathbb{C}^n$ is an irreducible curve and $f \in \mathbb{C}[V]$, then any poles of f must occur at points at infinity.

2. Splittings of Groups

Theorem 2.1 (Culler-Shalen)**.** *Let Γ be a finitely generated group. Assume that $C \subset X(\Gamma)$ is an irreducible curve with smooth projective model $\widetilde{C}$. Then associated to each point at infinity of $\widetilde{C}$ there is a non-trivial splitting of Γ.*

The key theorem we use is:

Theorem 2.2. *Assume that $\rho : \Gamma \to \mathrm{SL}_2(F)$ is a representation where F is a field with discrete valuation ν and there exists $\gamma \in \Gamma$ such that $\nu\Big(\mathrm{Tr}\big(\mathcal{P}(\gamma)\big)\Big) < 0$. Then there exists a non-trivial splitting of Γ.*

claim: Suppose we are in the case of $\widetilde{C}$ as in Theorem 2.1. If $\gamma \in \Gamma$, then TFAE:

1. $I_\gamma(\chi) \in \mathbb{C}$ (i.e. I_γ does not have a pole at x);
2. γ is conjugate into a vertex stabilizer

Pf. of claim:

$$\begin{aligned} I_\gamma(\chi) \in \mathbb{C} &\iff v_x(I_\gamma) \geq 0 \\ &\iff v_x(\mathrm{Tr}(P(\gamma))) \geq 0 \text{ since } \mathrm{Tr}(P(\gamma)) = I_\gamma \\ &\iff \mathrm{Tr}(P(\gamma)) \in \mathcal{O}_{\chi,\widetilde{C}} \\ &\iff P(\gamma) \text{ is conjugate into a vertex stabilizer} \end{aligned}$$

$\diamond$

Proof of Theorem 2.1. Consided $D \subset R(\Gamma)$ such that $t(D) = C$. Write $\Gamma =< a_1, \ldots, a_n | R_1, \ldots >$. We have $R_0(\Gamma) \subset R(\Gamma)$ and

$$\rho(a_1) = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \text{ and } \rho(a_2) = \begin{pmatrix} a_2 & 0 \\ c_2 & d_2 \end{pmatrix}$$

Pass to the smooth models:

$$\begin{array}{ccc} \widetilde{D} & \longrightarrow & D \\ \downarrow t & & \downarrow t|D \\ \widetilde{C} & \longrightarrow & C \end{array}$$

Note:

(1) $P : \Gamma \to \mathrm{SL}_2(\mathbb{C}(\widetilde{D}))$
(2) $t^{-1}(\widetilde{x}) = \{\widetilde{y_1}, \ldots, \widetilde{y_k}\}$ is a collection of points at infinity of $\widetilde{D}$.

Thus consider $\widetilde{y} \in \widetilde{D}$ with $t(\widetilde{y}) = \widetilde{x}$. We now apply the splitting theorem with $\mathbb{C}(\widetilde{D})$ and $\nu_{\widetilde{y}}$, the field F and the valuation ν. To get the nontrivial splitting we need to ensure that there exists $\gamma \in \Gamma$ such that $\nu_{\widetilde{y}}(\mathrm{Tr}(P(\gamma))) < 0$.

For all $\gamma \in \Gamma$, $I_\gamma : C \to \mathbb{C}$ is a polynomail function. If $\{\chi_{\rho_n}\} \subset C$ is a sequence of characters $\chi_{\rho_n} \to \widetilde{X}$, then $I_\gamma(\chi_{\rho_n}) = \chi_{\rho_n}(\gamma)$ so there exists γ such that

$$|\chi_{\rho_n}(\gamma)| \to \infty \implies |I_\gamma(\chi_{\rho_n})| \to \infty$$

But this means that I_γ has a pole at $\widetilde{x}$. Hence the splitting is nontrivial, as needed. $\square$

Remark. We say that a sequence of characters $\{\chi_{\rho_n}\}$ **blows up** if there exists a γ such that $|\chi_{\rho_n}(\gamma)| \to \infty$.

The mantra is that vertex stabilizers have integral traces since they are conjugate into $\mathrm{SL}_2(\mathcal{O}_\nu)$. Note that $\nu\Big(\mathrm{Tr}\big(\mathcal{P}(\gamma)\big)\Big) < 0$ iff I_γ has a pole on $\widetilde{C}$. This can be rephrased as the following:

Key point: $\gamma \in \Gamma$ lies in a vertex stabilizer iff

$$\begin{aligned} v_{\widetilde{x}}(I_\gamma) \geq 0 &\iff \mathrm{Tr}(P(\gamma)) \in \mathcal{O}_{\widetilde{x},\widetilde{C}} \\ &\iff I_\gamma \text{ does not blow up at } \widetilde{x} \end{aligned}$$

2.1. March 4: Applications to 3-manifolds.

Theorem 2.3. *Let $M = \mathbb{H}/\Gamma$ be a finite volume, orientable, cusped hyperbolic 3-manifold. Then Γ admits a non-trivial splitting.*

Proof. There exists a canonical component $X_0 \subset X(\Gamma)$ with $\dim X_0 = n =$ number of cusps of M. Hence we may apply Theorem 2.1. □

This is not very interesting since, for example, $K \subset \mathrm{S}^3$ (nontrivial knot) implies that $\pi_1(\mathrm{S}^3 \setminus K) \twoheadrightarrow \mathbb{Z}$ defines an HNN extension. How do we get other splittings of Γ?

For 3-manifold groups, finding splittings of Γ is equivalent to finding orientable, properly embedded, incompressible surfaces in M^3 with $\pi_1(M^3) = \Gamma$. In other words, for 3-manifolds, finding a nontrivial splitting of its fundamental group is equivalent to finding a system of essential surfaces in the manifold.

Definition 21. *$f : (S, \partial S) \to (M, \partial M)$ is **essential** if S is properly embedded, incompressible, and not boundary parallel.*

examples:

1. If K is the trefoil (note: a torus knot, so not hyperbolic), take S to be the black surface, which is not orientable. Then
$$\pi_1(\mathrm{S}^3 \setminus K) = <\pi_1(S), t | R_1, \dots>$$
2. swallow-follow torus in connect sums (also not hyperbolic)
3. 4-punctured spheres

These surfaces give rise to splittings of Γ. Consider the following:

Notation: associating a surface to an action Suppose M is a compact, orientable, irreducible 3-manifold with nonempty boundary. Then ∂M will consist of disjoint union of incompressible tori. Let $p : \widetilde{M} \to M$ be the universal cover for M. Assume that $\pi_1(M)$ acts non-trivially and without inversions on a tree T. Let E be the set of midpoints of all edges of T.

Definition 22. *A surface S is **associated to the action** if there exists a $\Gamma = \pi_1(M)$-equivariant map*
$$\widetilde{\varphi} : \widetilde{M} \to T$$
(i.e. $\widetilde{\varphi}(\gamma.x) = \gamma.\widetilde{\varphi}(x)$ such that $\widetilde{\varphi} \pitchfork E$ and $p^{-1}(S) = \widetilde{\varphi}^{-1}(E)$).

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{\varphi}} & T \\ \downarrow p & & \\ M & & \end{array}$$

We have the following classical theorem

Theorem 2.4. *If the action of $\pi_1(M)$ is non-trivial and acts without inversions then there exists a non-empty, essential surface $S \subset M$ which is associated to the action. Furthermore, if $C \subset \partial M$ is such that $\pi_1(C) \hookrightarrow \mathrm{Stab}_v\big(\pi_1(M)\big)$, then S may be chosen to be disjoint from C.*

Theorem 2.5. *Assume that M is as above and that $X(\pi_1 M)$ contains a curve. Then M is Haken.*

> CONJECTURE: If M is a closed, hyperbolic 3-manifold then M has a finite sheeted cover which is Haken.
>
> This leads to the question: With same hypotheses, does M have a finite sheeted cover N so that a component of $X(N)$ has dimension bigger than zero?

Corollary 2.6. *If M is irreducible and not Haken, then $X(M)$ is a finite set of points.*

Fix M as usual and also assume that ∂M is a torus T.

Definition 23. *A **slope** s on T is an unoriented isotopy class of simple closed curves on T.*

We usually fix a basis $\langle \mu, \lambda \rangle$ for $\pi_1(T)$ and thus identify the set of slopes on T with $\mathbb{Q} \cup \{\infty\}$. If $(S, \partial S)$ is an essential surface in M with non-empty boundary, then ∂S is a finite collection of simple, closed curves on T all with a common slope s.

Definition 24. *A **boundary slope** is such a slope.*

Theorem 2.7 (Hatcher)**.** *Take M as above. Then the set of all boundary slopes on M is finite.*

We say that an essential surface in M is detected by the Culler-Shalen machine if it arises as a surface associated to the action given by an ideal point of $X(M)$.

If S is such a surface with boundary slope $c = p/q$, so $c = \mu^p \lambda^q$. Let $R_0 \subset R(\Gamma)$ be such that $t(R_0) = X_0$ and $\dim R_0 = 1$. Conjugate ρ so that points of R_0 have

$$\rho(\mu) = \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix} \text{ and } \rho(\lambda) = \begin{pmatrix} y & * \\ 0 & y^{-1} \end{pmatrix}$$

Let $\widetilde{y} \in \widetilde{R_0}$ be a point at infinity with $t(\widetilde{y}) = \widetilde{x}$.
note: $c = \mu^p \lambda^q$ lies in a vertex stabilizer. This in turn implies that $I_{\mu^p\lambda^q}$ does not have a pole at the relevant ideal point. If ν is our valuation on $\mathbb{C}(R_0)$ this means that $\nu(I_{\mu^p\lambda^q}) \geq 0$.

Recall our tautological representation $\mathcal{P} : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C}(R_0))$. We may conjugate in $\mathrm{SL}_2(\mathbb{C}(R_0))$ to assume that

$$\mathcal{P}(\mu) \ = \ \begin{pmatrix} M & * \\ 0 & M^{-1} \end{pmatrix} \qquad \text{and} \qquad \mathcal{P}(\lambda) \ = \ \begin{pmatrix} L & * \\ 0 & L^{-1} \end{pmatrix}.$$

Then

$$\mathcal{P}(\mu^p\lambda^q) \ = \ \begin{pmatrix} M^pL^q & * \\ 0 & M^{-p}L^{-q} \end{pmatrix}.$$

So,

$$\nu\big(I_{\mu^p\lambda^q}\big) \ = \ \nu\big(M^pL^q + (M^pL^q)^{-1}\big) \geq 0.$$

Using a basic fact about valuations, and the fact that the trace must be integral, this means that

$$0 \;=\; \nu(M^p L^q) \;=\; p\nu(M) + q\nu(L)$$

Therefore,

$$\frac{p}{q} \;=\; -\frac{\nu(L)}{\nu(M)}.$$

The important thing to note here is that we may compute the boundary slopes from the valuation of the basis elements of $\pi_1(T)$.

2.2. **March 9: More 3-manifold applications.** Recall:

$X(\Gamma) \supset$ curve	$\rightsquigarrow$	splitting of Γ associated to $\widetilde{x} \in \widetilde{C}$	$\rightsquigarrow$	essential surface $S \subset M$ associated to the action of Γ on T_{ν_x} (detected by C-S machinery)

S has $\partial S = \emptyset$ or $\partial S = \mathbb{T}^2$ (inclusion) and ∂-slope should be able to be computed from valuations. Let M be compact, orientable, connected, and irreducible as usual. $\Gamma = \pi_1(M)$.

Lemma 2.8. *Assume that ∂M is an incompressible torus. Let C be a curve in $X(\Gamma)$ and $\widetilde{x}$ a point at infinity for C. Assume that there exists $\alpha \in \pi_1(T)$ such that $I_\alpha(\widetilde{x}) \in \mathbb{C}$. Then either*

- *$I_\beta(\widetilde{x}) \in \mathbb{C}$ for every element $\beta \in \pi_1(T)$, or*
- *α is a boundary slope.*

Proof. Note: $P(\alpha)$ is conjugate into a vertex stabilizer by hypothesis. The CS machine provides an essential surface S associated to the action on T_{ν_x}. If $\partial S \neq \emptyset$ then α is a ∂-slope. This is true since from EC's discussion, the associated surface S can be made disjoint from a simple closed curve C on $\mathbb{T}^2$ parallel to α. If α were not the ∂-slope, then there would exist a $\beta \in \mathbb{T}^2$ which is the ∂-slope, but β meets C, a contradiction. □

Theorem 2.9. *Take M as in the lemma, or at worst $\partial M = \emptyset$. If $X(M)$ contains a component of dimension bigger than one, then M contains a closed, essential surface.*

Definition 25. *M as above is* ***small*** *if M does not contain a closed essential surface.*

Corollary 2.10. *M small implies that all components of $X(M)$ have dimension less than 2.*

Corollary 2.11. *If $M = \mathrm{S}^3 \setminus K$ where K is a 2-bridge knot, then $X(\mathrm{S}^3 \setminus K)$ has only components of dimension less than 2.*

Proof. M small by Hatcher-Thurston, so apply Corollary 2.11. □

Note that there do exist such 3-manifolds with every component having dimension less than two, and containing a closed essential surface. M_{137} is such an example.

Proof of Theorem 2.9. Assume that $\partial M = \mathbb{T}^2$, otherwise done. By Hatcher's theorem (2.7), there exist only finitely many boundary slopes. Let $\beta \in \pi_1(\mathbb{T})$ be a non-∂-slope. Let $X \subset X(M)$ be a component of dimension $n \geq 2$. Then $I_\beta : X \to \mathbb{C}$ is a polynomial function. We can find $t \in C$ such that $I_\beta(t)^{-1} \neq \emptyset$. Then I_β^{-1} is an algebraic set whose irreducible components have dimension at least 1. Thus there exists at least a curve on which I_β is constant. Let $C \subset I_\beta^{-1}(t)$ be a curve and consider $\widetilde{x} \in \widetilde{C}$ a point at infinity. By Lemma 2.8, since β is not a boundary slope, we deduce that $I_\gamma(\widetilde{x}) \neq \infty \ \forall \ \gamma \in \pi_1(\mathbb{T}^2)$, which implies that there exists a closed essential surface in M. □

Theorem 2.12. *Let M be $\mathbb{H}/\Gamma$, finite volume, and $\partial M = T_1^2 \amalg T_2^2$. Assume that M is small. Then there exists an essential surface $S \subset M$ such that $\partial S \cap T_1 \neq \emptyset$ but $\partial S \cap T_1 = \emptyset$.*

Proof. We have $\dim X_0 = 2$ where $X_0 \subset X(M)$ is the canonical component. Define for $\alpha \in \pi_1(\mathbb{T}_2)$ the map $f_\alpha : X_0 \to \mathbb{C}$ by $f_\alpha(\chi_\rho) = \chi_\rho^2(\alpha) - 4$. Note that at the faithful discrete representation ρ_0, $f_\alpha(\chi_{\rho_0}) = 0 \; \forall \; \alpha \in \pi_1(\mathbb{T}_2)$. Fix attention on $\alpha_0 \in \pi_1(\mathbb{T}_2)$ and consider $Y = f_\alpha^{-1}(0)$. Then Y is an algebraic set all of whose irreducible components have dimension at least 1. If dim $= 2$ then $X_0 = \{Y = 0\}$ (?), but Dehn surgery results tell us that $\chi_\rho(\alpha)$ is nonconstant on X_0, a contradiction.

Then all components of Y have dimension 1. Note: if $f_\beta : C \to \mathbb{C}$ satisfies

$$f_\beta(\chi_\rho) = 0 \; \forall \; \chi_\rho \tag{2}$$

then we are done since in this case, choosing an irreducible component $C_1 subset C$ and $\widetilde{x} \in \widetilde{C_1}$, we obtain an essential surface that is disjoint from T_2. Finally, M small implies that $\partial S \cap T_1 \neq \emptyset$. Thus condition 2 holds, since performing Dehn surgery on T_1 while keeping T_2 complete gives uncountably many points at which $f_\beta(\chi_\rho) = 0$ as required. □

Definition 26. *$P < \Gamma$ is a peripheral subgroup, then $\varphi : \Gamma \to \pi_1 N$ is* ***peripheral preserving*** *if $\varphi(P) <$ a peripheral subgroup of $\pi_1 N$.*

Remark. N can be closed, in which case $\varphi(P) = 1$.

Theorem 2.13. *Let M be $\mathbb{H}/\Gamma$, finite volume, with one cusp, and small. Then there exist only finitely many distinct, finite volume, hyperbolic 3-manifolds, N_i, for which there exists a peripheral preserving epimorphism $\varphi_i : \Gamma \to \pi_1(N_i)$.*

Proof. Assume there is a sequence $\{N_j\}$ of distinct hyperbolic 3-manifolds such that there exists

$$\Gamma \xrightarrow{f_j} \pi_1 N_j \xrightarrow{\rho_j} \mathrm{SL}_2(\mathbb{C})$$

which is peripheral preserving. Let $\rho_j : \pi_1 N_j \to \mathrm{SL}_2(\mathbb{C})$ be faithful and discrete. Let $\varphi_j = \rho_j \circ f_j : \Gamma \to \mathrm{SL}_2(\mathbb{C})$. The characters χ_{φ_j} are distinct characters on $X(M)$. By passing to a subsequence, we may assume that $\{\chi_{\varphi_j}\} \subset C$ where C is an irreducible curve. Note that $f_j(P)$ peripheral implies that $\chi_{\varphi_j}(\alpha) = \pm 2 \; \forall \; \alpha \in P$.

If we pass to a point at infinity, say $\widetilde{x} \in \widetilde{C}$, we have $I_\alpha(\widetilde{x}) \neq \infty \; \forall \; \alpha \in P$. Hence we obtain a closed essential surface in M, a contradiction. □

2.3. **March 11: Essential Surfaces from actions on trees.** Assume that M is compact, orientable, and irreducible, and let $\Gamma = \pi_1(M), p : \widetilde{M} \to M$ be the universal cover. Fix a triangulation on M and lift it to $\widetilde{M}$. Let E denote the set of midpoints of edges in a tree T, and suppose we have a simplicial action of Γ on T. Let E be the set of all midpoints of edges in the tree T.

Note: If $f : \widetilde{M} \to T$ is a Γ-invariant simplicial map, then $\widetilde{S} = f^{-1}(E)$ is a properly embedded surface in $\widetilde{M}$. Furthermore, $\widetilde{S}$ is invariant under the action of Γ on $\widetilde{M}$. Therefore, $S = p(\widetilde{S})$ is a properly embedded surface in M.

Definition 27. *A properly embedded surface $S \subset M$ is **associated to the action** if there exists a Γ-invariant map $f : \widetilde{M} \to T$ such that*

$$p^{-1}(S) \;=\; f^{-1}(E).$$

Proposition 2.14. *There exists a surface associated to the action of Γ on T.*

Proposition 2.15. *There exists a simplicial Γ-invariant map $f : \widetilde{M} \to T$.*

Proof. Let $S^{(0)}$ be a complete set of orbit representations for Γ-action on $\widetilde{M}^{(0)}$. Define $f^{(0)} : S^{(0)} \to T^{(0)}$ by $f^{(0)}(\gamma\dot{x}) = \gamma\dot{f}^{(0)}(x)\ \forall\ \gamma \in \Gamma, x \in S^{(0)}$ where $f^{(0)} : \widetilde{M}^{(0)} \to T^{(0)}$. Assume $f^{(i)} : \widetilde{M}^{(i)} \to T$ is Γ-invariant and simplicial. We want $f^{(i+1)} : \widetilde{M}^{(i+1)} \to T$. First extend on a complete set of orbit representations of the action of Γ on $(i+1)$-complexes and extend by Γ-invariance. $\square$

Proposition 2.16. *If S is associated to the action, then for each component C_i of $M - S$, $\pi_1(C_i)$ stabilizes a vertex of T.*

Proof. Since $\widetilde{S} = p^{-1}(S)$ is Γ-invariant, Γ acts on the components of $\widetilde{M} \setminus \widetilde{S}$ and Γ_i stabilizes a component $\widetilde{C}$ of $p^{-1}(C_i)$, which implies that

$$f(\widetilde{C}) = f(\Gamma_i \cdot \widetilde{C}) = \Gamma_i \cdot f(\widetilde{C})$$

Thus $f(\widetilde{C}) \subset T \setminus E$, so Γ_i stabilizes the corresponding vertex. $\square$

Corollary 2.17. *If $S = \emptyset$, then Γ stabilizes a vertex of T. (The action of Γ on T is trivial.)*

Proposition 2.18. *If Γ acts on T without inversions, then S is orientable.*

Proof. We'll prove the contrapositive. Assume there exists a component of S that is not orientable. Then, for some component $\widetilde{S_0}$ of $\widetilde{S} = p^{-1}(S)$, there exists a $\gamma \in \Gamma$ that interchanges two components C_1, C_2 or $\widetilde{M} \setminus \widetilde{S}$ that are separated by $\widetilde{S_0}$. But $\widetilde{S_0}$ is connected, so $f(\widetilde{S_0}) =$ midpoint of an edge in T implies that Γ does not act without inversions. $\square$

Proposition 2.19. *If C is a connected submanifold of ∂M such that the image of $\pi_1(C)$ in Γ is contained in a vertex stablizer, then the surface may be taken to be disjoint from C.*

Proof. Build a Γ-invariant submanifold of ∂M such that $p(f^{-1}(E)) \cap C \neq \emptyset$. Let $\widetilde{C}$ be a component of $p^{-1}(C)$. Choose the triangulation such that C is a sub-complex. When choosing $S^{(0)}$, choose representations from $\widetilde{C}^{(0)}$ whenever possible. To define $f^{(0)} : S^{(0)} \to T^{(0)}$, if $p \in S^{(0)} \cap \widetilde{C}^{(0)}$, then set $f^{(0)}(p) = v$. Note: if $\gamma \in \Gamma_i$, then $f^{(0)}(\gamma \cdot p) = \gamma f^{(0)}(p) = \gamma \cdot v = v$, so $f^{(0)}(\widetilde{C}^{(0)}) = v$. By construction, 0-skeletons of translates of $\widetilde{C}$ are sent to translates of v, that is, $f(\widetilde{C}) = v$, so $p^{-1}(C) \cap f^{-1}(E) = \emptyset$, so $p(f^{-1}(E)) \cap C = \emptyset$. $\square$

Definition 28. *An **essential surface** S in M is a properly embedded surface which satisfies*

1. *$S \neq \emptyset$.*
2. *S is orientable.*
3. *S has no compressing discs.*
4. *S has no sphere components.*
5. *S has no boundary parallel components.*

Theorem 2.20. *There is an essential surface in M associated to the action of Γ on T. Furthermore, if C is a connected submanifold of ∂M such that the image of $\pi_1(C)$ in Γ is contained in a vertex stablizer, then the surface may be taken to be disjoint from C.*

Proof. Assume $S = p(f^{-1}(E))$ as usual, where S has a compressing disk D in M. Let B be a regular neighborhood of D, so B is a 3-ball, $B \cap S = A$ is an annulus, X_- is a solid torus, and X_+ is a 3-ball. Let $\widetilde{B}$ be a component of $p^{-1}(B)$. Then $f^{-1}(E) \cap \widetilde{B} = \widetilde{A}$. Also, $f(\widetilde{X}_\pm)$ map into 2 adjacent connected components of $T \setminus E$, say $Y_\pm$. D_1 and D_2 separate $\widetilde{B}$ into 3 3-balls, B_1, B_2, B_3. Define

$$g : \widetilde{B} \to T,\ g|_{\partial \widetilde{B}} = f|_{\partial \widetilde{B}},\ g(D_1 \cup D_2) = f(\widetilde{A})$$

Notice that $g(\partial B_2) \subset Y_- \cup f(\widetilde{A}),\ g(\partial B_i) \subset Y_+ \cup f(\widetilde{A})$ for $i = 1, 3$. Extend g to $\widetilde{B}$ such that $g(\text{int} B_2) \subset Y_-,\ g(\text{int} B_i) \subset Y_+$ for $i = 1, 3$. Finally, extend g to $p^{-1}(B)$ by Γ-invariance, and extend to $\widetilde{M}$ with f. $\square$

2.4. **March 23: More 3-manifold stuff.** Recall: $\Gamma = \pi_1(M^3)$, M^3 compact, orientable, irreducible. If $X(\Gamma)$ contains a curve C, then by passing to points at $\infty, \widetilde{x} \in \widetilde{C} \setminus C$, we obtain an action of Γ on $T_{\widetilde{x}}$ and this action determins a splitting of Γ. We can then obtain essential surfaces in M "dual to the action".

Define $\gamma \in \Gamma$, $I_\gamma : C \to \mathbb{C}$, $I_\gamma(\chi_\rho) = \chi_\rho(\gamma)$. If we consider M as above, then $\partial M = \mathbb{T}^2$ is incompressible.

example: If $\alpha \in \pi_1(\mathbb{T}) \leq \Gamma$ and $I_\alpha(\widetilde{x}) \neq \infty$, then either α is a boundary slope or $I_\beta(\widetilde{x}) \neq \infty \ \forall \ \beta \in \pi_1(\mathbb{T}^2)$. In the latter case, this implies that M contains a closed essential surface.

Recall that if M is finite volume, hyperbolic with one cusp, then I_α is non-constant on X_0 for all peripheral α.

Theorem 2.21. *Let $M = \mathbb{H}/\Gamma$ have finite volume and one cusp. Then M has at least two boundary slopes.*

Lemma 2.22. *If s is any slope on $\partial M = \mathbb{T}^2$, then there is an essential surface $\Sigma \subset M$ with $\partial\Sigma \neq \emptyset$ and has boundary slope not equal to s.*

Proof. Let $\gamma \in \Gamma$ represent s. Consider $I_\gamma : X_0 \to \mathbb{C}$. Then I_γ is nonconstant and therefore has poles which must arise at points at infinity for X_0. Choose such a point $\widetilde{x} \in \widetilde{X_0} \setminus X_0$. Hence there is an essential surface $\Sigma \subset M$, associated to the action on $T_{\widetilde{x}}$. From the remarks above, $\partial\Sigma \neq \emptyset$. The boundary slope of $\Sigma \neq s$ since I_γ is blowing up at $\widetilde{x}$. $\square$

examples:

(1) The figure eight has boundary slopes $\frac{0}{1}, \pm\frac{4}{1}$ (Hatcher-Thurston)
(2) All twist knots (except the trefoil and the unknot) have three boundary slopes.
(3) The James Bond manifold (M_{007}) has exactly two detected slopes.

Question: Is there any special property of the surfaces detected by the C-S machinery?

Theorem 2.23. *Assume that $M = \mathbb{H}/\Gamma$ has finite volume and one cusp. Then the Culler-Shalen machine does not detect fibers (or virtual fibers) on components of $X(\Gamma)$ which contain an irreducible representation.*

Notation: Let S be an orientable surface, $|\partial S| = 1, \Theta : S \to S$ a self-homeomorphism. Form a 3-manifold $M_\Theta = (S \times I)/\sim$. Then M_Θ fibers over S^1 with fiber $S \subset M_\Theta$ is incompressible. This defines a short exact sequence:

$$1 \to \pi_1 S \to \pi_1 M_\Theta \to \mathbb{Z} \to 1$$

Proof of Theorem 2.23. $\Gamma = \pi_1 M, \pi_1 S \langle\langle \Gamma \rangle\rangle$. The claim is that for any point at infinity $\widetilde{x} \in \widetilde{X_0}$, an essential surface associated to the action is never a fiber. We assume to the contarary that S is detected by passing to such an $\widetilde{x}$. Therefore $P(\pi_1 S) \subset$ vertex stabilizer if and only if $v_{\widetilde{x}}(I_\gamma) \geq 0 \ \forall \ \gamma \in \pi_1 S$, which is equivalent to $\mathrm{Tr}P(\gamma) \in \mathcal{O}_{\widetilde{x}, \widetilde{X_0}}$. We require a lemma:

Lemma 2.24. *Let F be a field with a valuation ν. Let G be a subgroup of $\mathrm{SL}_2(F)$ and assume that there exists $N \triangleright G$ (non-central) such that N is $\mathrm{GL}_2(F)$-conjugate* *into* $\mathrm{SL}_2(\mathcal{O})$. *Then either* G *contains a solvable normal subgroup, or*

$$\mathrm{Tr}(g) \in \mathcal{O} \qquad \forall \;\; g \in G.$$

Proof. Outline: Assume $g \in G$ such that $\mathrm{Tr}(g) \notin \mathcal{O}$. Extend F and ν so that we can write g as a diagonal matrix. Take $\eta \in N$. Then $\mathrm{Tr}(\eta g^n \eta g^{-n} \in \mathcal{O}$ for every n. This implies that η is either upper or lower triangular. Hence N is either upper or lower triangular. Therefore, N is solvable.

Now assume that there is $g \in G$ such that $\mathrm{Tr} g \notin \mathcal{O}$, so g has characteristic polynomial $\lambda^2 - (\mathrm{Tr} g)\lambda + 1 = 0$, where λ, λ^{-1} are the eigenvalues of g. Note by definition, $F(\lambda)$ is an extension of degree no more than 2 over F. Assume that $\lambda \in F$. Then we can conjugate G so g is of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

$N\langle\langle G\rangle\rangle$ implies that $g^n N g^{-n} = N \;\forall\; n \in \mathbb{Z}$. Let $\eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N$, and consider

$$g^n \eta g^{-n} = \begin{pmatrix} a & b\lambda^{2n} \\ c\lambda{-}2n & d \end{pmatrix} \in N$$

Therefore $\mathrm{Tr}(\eta g^n \eta g^{-n}) \in \mathcal{O}$, so

$$x = a^2 + d^2 + bc(\lambda^{2n} + \lambda{-}2n) \in \mathcal{O} \;\forall\; n \in \mathbb{Z}$$

But $x = (a+d)^2 - 2 + bc(\lambda^n - \lambda{-}n)^2$, so $x \in \mathcal{O}$ if and only if $bc(\lambda^n - \lambda^{-n})^2 \in \mathcal{O}$, which is the same as $v(bc(\lambda^n - \lambda^{-n})^2) \geq 0$, which is equivalent to $v(b) + v(c) + 2v(\lambda^n - \lambda^{-n}) \geq 0$. The key point to keep in mind is that $v(\lambda^n - \lambda^{-1}) \to -\infty$ as $n \to \infty$, a contradiction.

$\mathrm{Tr} g = \lambda + \lambda^{-1}$. Note that if $\mathrm{Tr} g \notin \mathcal{O}$, then $v(\lambda) \neq 0$, so $v(\mathrm{Tr} g) < 0$. By properties of valuations, we have

$$v(\lambda + \lambda^{-1}) = v(\frac{\lambda^2 + 1}{\lambda} = v(lambda^2) - v(\lambda) \geq 0$$

Hence $b = 0$ or $c = 0$, so $\forall\; \eta \in N$, either

$$\eta = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \text{ or } \eta = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$$

But both forms of above cannot occur since taking a product gives an element of N not of the above form, so N is of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \text{ or } \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

both of which are solvable.

(Note: the hypothesis "not central" was used because otherwise we would get no info from commuting matrices.) □

Continuing with Theorem 2.23, we let $t : R_0 \to X_0$ and $F = \mathbb{C}(R_0)$, where R_0 is a curve contained in $R(\Gamma)$. Then $P : \Gamma \hookrightarrow \mathrm{SL}_2(\mathbb{C}(\mathbb{R}_0))$.
note: R_0 contains a faithful, discrete representation. Hence $P : \Gamma \to \mathrm{SL}_2(\mathbb{C}(R_0))$ is faithful, for if it weren't, there would exist $1 \neq \gamma \in \Gamma$ such that

$$I = P(\gamma) = \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix}$$

and so $\rho(\gamma) = 1 \;\forall\; \rho \in R_0$, which is false. Hence $P\Gamma \cong \Gamma$ is a finite volume hyperbolic 3-manifold group, so it contains no normal solvable subgroup. Therefore

Lemma 2.24 applies to yield $\mathrm{Tr}g \in \mathcal{O}_{\widetilde{x},\widetilde{X_0}}$ $\forall\ g \in \Gamma$. But some I_γ is blowing up since the surface S is being detected. This contradiction completes the proof. □

QUESTION: Assume that $\Sigma \subset M = \mathbb{H}/\Gamma$ is one cusped and finite volume and $\partial\Sigma \neq \emptyset$. Assume also that the surface Σ is not a fiber or virtual fiber. Is Σ detected by the Culler-Shalen machine?

2.5. **March 25: The Smith Conjecture.**

Theorem 2.25. *Let Σ be a (homotopy) 3-sphere and $K \subset \Sigma$ a (tame) knot. Let $\widetilde{\Sigma_n}$ be the n-fold cyclic cover of Σ branched over K. If $\widetilde{\Sigma_n}$ is simply connected then K is the trivial knot.*

This theorem implies the Smith Conjecture (revised).

Theorem 2.26 (The Smith Conjecture)**.** *If $h : \widetilde{\Sigma} \to \widetilde{\Sigma}$ is a periodic diffeomorphism with non-empty fixed point set, then* $\mathrm{fix}(h)$ *is the trivial knot.*

Recall: an n**-fold branched cover** of Σ is $(\widetilde{\Sigma_n}, p)$:

$$\begin{array}{ccc} \widetilde{\Sigma_n} & \qquad & \widetilde{\Sigma_n} \setminus \widetilde{K} \\ \downarrow{\scriptstyle p} & & \downarrow \\ \Sigma \supset K & & \Sigma \setminus K \end{array}$$

$\widetilde{K} = p^{-1}(K)$ such that for all $x \in p^{-1}(K)$, there is a neighborhood homeomorphic to $D \times I$ such that p has the form $(z, t) \mapsto (z^n, t), n \geq 2$. Alternatively, there is a cyclic group of diffeomorphisms C acting on $\widetilde{\Sigma_n}$ with nonempty fixed point set $\widetilde{K}$ and $p(\widetilde{K}) = K$.

Take $\Sigma = \widetilde{\Sigma}/\langle h \rangle$ and $K = \mathrm{fix}(h)$. Then Σ is simply connected.

Proof. Proof of Theorem 2.25 We assume that $M = \Sigma \backslash K$ is hyperbolic. Define the orbifold $Q_n = (\Sigma, K)$ where K has cone angle $\frac{2\pi}{n}$. Then if μ is a meridian of K, then μ^n bounds a disc in Q_n.

$$\Gamma_n \;=\; \langle\, \pi_1(M) \,|\, \mu^n = 1 \,\rangle$$

So we have the exact sequence

$$1 \longrightarrow \pi_1(\widetilde{\Sigma_n}) \longrightarrow \Gamma_n \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 1$$

If $\widetilde{\Sigma_n}$ is simply connected then $\Gamma_n \cong \mathbb{Z}/n\mathbb{Z}$. Thus, our strategy is to show that Γ_n is not cyclic. We do this by finding a representation into $\mathrm{PSL}_2(\mathbb{C})$ with non-cyclic image.

We attempt to arrange a representation $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ such that

- $\rho(\mu)$ has order $2n$, and
- ρ descends to a representation $\overline{\rho} : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ with non-cyclic image.

This would be a win because in the following diagram, we would have $\overline{\rho}$ noncyclic but $\overline{\rho}(\mu)^n = 1$:

$$\begin{array}{ccc} \pi_1 M & \xrightarrow{\ \overline{\rho}\ } & \mathrm{PSL}_2(\mathbb{C}) \\ & \searrow \quad \nearrow & \\ & \Gamma_n & \end{array}$$

By assuming that M is hyperbolic, we are guaranteed that $I_\mu : \widetilde{X_0} \to \mathbb{C}P^1$ is surjective. Let ω be a primitive $2n$-th root of unity. Then there exists $\chi \in \widetilde{X_0}$ such that $I_\mu(\chi) = \omega + \omega^{-1} = t$.

If χ is a point at infinity, then either μ is a boundary slope or M contains a closed, essential surface. We apply the following.

Theorem 2.27 (Gordon, Litherland). *Let M be as above and assume that either μ is a boundary slope or M contains a closed, essential surface. Then any regular, branched cover of M contains a closed, essential surface. In particular, no regular, branched cover is simply connected.*

So now we may assume that $\chi \in X_0$.

The rest follows from the following lemma. It can be proved by counting dimensions and the fact that if $\overline{\rho}$ has cyclic image, then ρ is abelian. In fact,

$$\rho\big(\pi_1(M)\big) \;\cong\; \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Lemma 2.28. *Suppose $M = \mathbb{H}/\Gamma$ is orientable, 1-cusped, and finite volume. Then given any $\chi \in X_0$ there is a representation $\rho \in t^{-1}(\chi)$ such that $\mathrm{Im}(\overline{\rho})$ is not cyclic.*

Proof. Note that for most points on X_0, this is clear as the representations are irreducible representations into $\mathrm{SL}_2(\mathbb{C})$ (here, $t^{-1}(\chi) = \mathrm{SL}_2(\mathbb{C})$-orbit of given representation ρ with $t(\rho) = \chi$).

It follows from algebraic geometry that $t^{-1}(\chi)$ has dimension at least 3 for all $\chi \in X_0$. Consider $\chi \in X_0$ for which if $\rho \in t^{-1}(\chi)$, $\overline{\rho}(\pi_1 M)$ is cyclic. What does $\rho(\pi_1 M) \subset \mathrm{SL}_2(\mathbb{C})$ look like (if not cyclic)? We claim that it is abelian. Note that $-I \in \rho(\pi_1 M)$, otherwise $\overline{\rho}(\pi_1 M)$ not cyclic. Thus we have a short exact sequence

$$1 \to \{\pm 1\} \to \rho(\pi_1 M) \to \mathbb{Z}_n \to 1$$

so $\rho(\pi_1 M) = \mathbb{Z}_n \times \mathbb{Z}_2$, hence ρ is a diagonal representation. To complete the proof, it is an exercise to show that $\dim(B) \leq 2$ where B is the set of diagonal representations in $t^{-1}(\chi)$. □

□

3. The A-polynomial

3.1. March 30: Definition of the A-polynomial.

We define the A-polynomial for knots in S^3, although it can be defined for knots in appropriate manifolds. Let K be a knot in S^3, and set $M = \int N(K)$. Fix a framing $\{l, m\}$ for $\pi_1(T)$. Let $\Gamma = \pi_1(M)$. We will see that the two-variable polynomial that we define is "almost an invariant".

Define $R_U \subset R(\Gamma)$ to be

$$R_U \;=\; \{\, \rho \in R(\Gamma) \,|\, \rho(l) \text{ and } \rho(m) \text{ are upper triangular} \,\}.$$

Write

$$\rho(l) \;=\; \begin{pmatrix} L & * \\ 0 & L^{-1} \end{pmatrix} \qquad \text{and} \qquad \rho(m) \;=\; \begin{pmatrix} M & * \\ 0 & M^{-1} \end{pmatrix}.$$

Define the polynomial map ξ from the closed algebraic set R_U to $\mathbb{C}^2\backslash\{0\}$ by $\xi(\rho) = (L_\rho, M_\rho)$. If C is an irreducible component of R_U then the Zariski closure of $\xi(C)$ is a variety in $\mathbb{C}^2$. This has dimension 0 or 1, so throw away 0-dim'l components and only consider those $C \subset R_U$ that project to curves (i.e. dim 1).

Theorem 3.1 (Hartshorne). *A variety $Y \subset \mathbb{C}^2$ is a curve if and only if Y is a zero set of a non-constant, irreducible polynomial in $\mathbb{C}[x, y]$.*

So, for each component C of R_U which projects to a curve under ξ we associate an irreducible polynomial $F_C(L, M) \in \mathbb{C}[L, M]$. Given R_U this gives a finite list of distinct polynomials $\{F_{C_i}\}_{i=1}^n$.

Definition 29. *Define the **A-polynomial** for K as*

$$A_K(L, M) \;=\; \frac{1}{L-1} \cdot \prod_{i=1}^{n} F_{C_i}(L, M).$$

Remarks.

(1) $\Gamma \to \mathbb{Z} \to SL_2(\mathbb{C})$, $m \mapsto$ a generator, $l \mapsto 1$ implies that $L = 1$ for these representations. $\{L - 1 = 0\}$ defines a polynomial which arises from an abelian representation of Γ.

Proposition 3.2. *If K is the unknot then $A_K \;=\; 1$.*

Remark. We will really take the product $\prod_{i=1}^n F_i(L, M)$ where the F_i are *distinct* irreducible polynomials.

Proposition 3.3. *If K is hyperbolic then $A_K \;\neq\; 1$.*

Proof. Crucial fact is that $I_l : X_0 \to \mathbb{C}$ is contant. If $R_0 \subset R_U$ is such that $t : R_0 \to X_0$, then I_l nonconstant implies that L must be nonconstant on R_0. But then teh component R_0 does not project to $(L - 1)$ in $\mathbb{C}^2$ under ξ. □

In sum: the first follows, since dividing by $L - 1$ results in ignoring abelian representations. The second follows from the fact that $I_l : X_0 \to \mathbb{C}$ is non-constant.

Proposition 3.4. *If K is the (p, q)-torus knot then A_K is divisible by $LM^{pq} + 1$.*

QUESTION: Does every non-trivial knot have a non-trivial A-polynomial?

Thanks to Kronheimer/Mrowka, any non-trivial knot admits an irreducible $SU(2) \subset \mathrm{SL}_2(\mathbb{C})$ representation. Can these be deformed to give a curve?

In fact they can. Dunfield/Garoufalidis have shown that every non-trivial knot has a non-trivial A-polynomial.

Proposition 3.5. *$A_K(L,M)$ involves only even powers of M.*

Proof. Consider the map α:

$$\begin{array}{ccl} \Gamma & \to & \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{SL}_2(\mathbb{C}) \\ m & \mapsto & \text{gen} \mapsto -1 \mapsto -I \\ l & \mapsto & 1 \mapsto 1 \mapsto I \end{array}$$

(Take α to be the composition of the above maps.) If $\rho \in R_U$ then multiply it with the representation α where $\alpha(m) = -I$ and $\alpha(l) = I$. This is a homomorphism since

$$\begin{aligned} \rho_\alpha(g_1 g_2) = \alpha(g_1 g_2)\rho(g_1 g_2) &= \alpha(g_1)\alpha(g_2)\rho(g_1)\rho(g_2) \\ &= \alpha(g_1)\rho(g_1)\alpha(g_2)\rho(g_2) \end{aligned}$$

Note that $\rho_\alpha(m) = \begin{pmatrix} -\mu & * \\ 0 & -\mu^{-1} \end{pmatrix} \in R_U, \rho_\alpha(l) = \rho(l)$. Thus $\xi(\rho\alpha) = (L, -M)$. ☐

Let Y_C be a one dimensional component of $\overline{\xi(C)}$ for some component $C \subset R_U$ and let $\widetilde{Y_C}$ be its smooth, projective model. Let $x \in \widetilde{Y_C} \backslash Y_C$.

$$\begin{array}{ccccc} \widetilde{X_C} & \longleftarrow & \widetilde{C} & \xrightarrow{\widetilde{\xi}} & \widetilde{Y_C} \\ \uparrow & & \uparrow & & \uparrow \\ X_C & \longleftarrow & C & \xrightarrow{\xi} & Y_C \end{array}$$

Consider a sequence of points $\{(L_n, M_n)\} \subset$ which converges to x. Then at least one of L_n, M_n, $1/L_n$, or $1/M_n$ approaches infinity in norm. This corresponds to a sequence of representations in $R(\Gamma)$ which converge to a point at infinity. Furthermore, the character of some elements of $\pi_1(T)$ are unbounded here. Therefore, the CS-machine gives us an essential surface in $\mathrm{S}^3 \backslash K$ with non-empty boundary. *Therefore, the beauty of the A-polynomial is that it only sees stuff that blows up at the boundary.*

Conversely, if $\{\chi_{\rho_n}\}$ is a sequence of charts for which $\chi_{\rho_n} \to y \in \widetilde{X_C} \setminus X_C$ that detect an essential surface with nonempty boundary, then projecting to Y_C we get a sequence of points $\{(L_n, M_n)\} \to$ a point at infinity of $\widetilde{Y_C} \backslash Y_C$. Otherwise, values of L_n, M_n stay bounded, which implies that characters stay bounded, a contradiction.

Summarizing:

Theorem 3.6. *Assume A_K is non-trivial. An essential surface with non-empty boundary is detected by the character variety if and only if it is detected by the A-polynomial.*

Suppose $F(L, M) \mid A_K(L, M)$ is an irreducible factor, and let $x \in \widetilde{Z}(F) \setminus Z(F)$. Associated to x is a valuation ν on $\mathbb{C}(V(F))$ and:

Corollary 3.7. *The boundary slope of the essential surface Σ detected by a point $x \in \widetilde{Y_C} \backslash Y_C$ is $-\frac{\nu(L)}{\nu(M)}$.*

3.2. **April 1: Newton polygons and Holes. Remark.** Holes: Assume $C \subset R(U)$ is a curve and $\overline{\xi(C)} \backslash \xi C$ is non-empty (a finite set of points). These are "holes". They are secretly like points at infinity in the following sense: if $(L, M) \in \overline{\xi(C)} \setminus \xi(C)$, then there does not exist a $\rho \in C$ such that $\xi(\rho) = (L, M)$. Thus, (L, M) is a hole. Now consider a sequence of points $(L_n, M_n) \to (L, M)$ (in the usual topology). Back in C we have a sequence of representations $\{\rho_n\}$ with $\xi(\rho_n) = (L_n, M_n)$. Note that each ρ_n is bounded on T. $\chi_{\rho_n}(\alpha)$ cannot be bounded for all $\alpha \in \pi_1 M$; otherwise, we could construct $\rho_n \to \rho$ a representation and $\xi(\rho) = (L, M)$, a contradiction. So we have a sequence of characters blowing up but bounded on $\pi_1 T$, so we get a closed essential surface. If they exist we cannot quite say "the A-polynomial only detects surfaces with non-empty boundary".

QUESTION: Do holes exist?

Let $P(x, y) \in \mathbb{C}[x, y]$. Define the Newton Polygon,

$$\text{Newt}(P) = \text{Convex hull in } \mathbb{R}^2 \text{ of } (i, j) \text{ where the coefficient of } x^i y^j \text{ is non} - \text{zero.}$$

examples:

(1) $f(x, y) = x^2(2-y)+(y^2-y-1)$, then $\text{Newt}(f)$ is the quadralateral with vertices at $(0, 0), (2, 0), (2, 1), (0, 2)$.
(2) $A_{4_1} = -M^4 + (1 - M^2 - 2M^4 - M^6 + M^8)L - M^4L^2$, then $\text{Newt}(A_{4_1})$ is the diamond with vertices at $(1, 0), (2, 4), (1, 8), (0, 4)$.
(3)

(3)

$$A_{5_2} = 1+(-1+2M^2+2M^4-M^8+M^10)L+(M^4-M^6+2M^10+2M^12-M^14)L^2+M^14L^3$$

imply that the slopes of $\text{Newt}(A_{5_2})$ are $\frac{0}{1}, \frac{4}{1}, \frac{10}{1}$, and $\text{Newt}(A_{5_2})$ is the hexagon with vertices at $(0, 0), (1, 0), (2, 4), (3, 14), (2, 14), (1, 10)$.

Theorem 3.8. *The slope of the boundary edges of* $\text{Newt}(A_K)$ *are boundary slopes of* K.

Outline of Proof. We must show that going to infinity on V_K, we associate an edge of Newt. Here are the main ideas:

Think about the example 5_2: going to a point at infinity, at least one of $M, L, \frac{1}{M}, \frac{1}{L}$ is blowing up. For example $M \to \infty$ but L bounded. Or consider A_{5_2} given by equation 3 and factor as a polynomial over M and divide through by M^14. Then, when $|M|$ is large, the resulting equation is basically $L^3 - L^2 = 0$ and we can associate an edge e to the degeneration. This phenomena is what happens more generally, i.e. going to infinity picks out an egde of Newt. $\square$

examples:

(1) $4_1 : \frac{\pm 4}{1}$ are boundary slopes
(2) $5_2 : \frac{0}{1}, \frac{4}{1}, \frac{10}{1}$ are boundary slopes
(3) $8_2 0 : \frac{-10}{1}, \frac{0}{1}, \frac{8}{3}$ is fibered, but $\frac{0}{1}$ is the boundary slope of an essential separating surface with nonempty boundary.

Newton polygon detects **all** boundary slopes. Hatcher and Thurston proved that all boundary slopes of 2-bridge knots are integral, and 8_{20} is the first alternating knot with non-integral boundary slope.

3.3. **April 6: Boundary Slopes are ∂-slopes. Remark.** Suppose $(X,)) \in V(A_K), X \neq 0$.

If Newt(A_K) has an edge with slope zero on the L-axis. Associate to it a surface of slope $\frac{0}{1}$ detected by the C-S machine.

example: $A_K(\frac{1}{L}, \frac{1}{M}) = \pm L^s M^t A_K(L,M)$ implies there are precisely two edges of a given slope s on Newt. The A-polynomial for K has the form $A_K(L,M) = \sum \alpha_\beta L^\beta + A(L,M)$ where α_β are constants and at least two of the $\alpha_\beta \neq 0$ and M divides every term of $A(L,M)$, that is, $A(L,0) = 0 \ \forall \ L$. Hence, $A_K(L,0) = \sum \alpha_\beta L^\beta \neq 0$ and so this has roots (by the edge assumption), that is we find at least one point $x = (r,0) \in V(A_K)$. (Recall: If a surface is detected of slope p/q at a point at infinity x, then $\frac{p}{q} = \frac{-V(L)}{V(M)}$. Let v_x be the valuation associated to x. Then $v_x(L) = 0$ since there is no zero or pole at this point and $v_x(M) > 0$ since M has a zero at x. By the above remark, $\frac{0}{1}$ is a detected slope.

claim: This suffices to show that all slopes of Newt(A_K) are detected. The idea is to change basis.

Given $(m,l) = \pi_1(T)$, compute $A_K(L,M)$. Now assume that $(m',l') = \pi_1(T)$ and compute $A'_K(L',M')$. We can interpolate between Newt(A_K) and Newt(A'_K) using the change of basis given by $(m,l) \leftrightarrow (m',l')$.

example: Figure 8 knot. $A_K(L,M) = -M^4 + (1 - M^2 - 2M^4 - M^6 + M^8)L - M^4L^2$. Make the change of coordinates: $M' = M^4L, L' = M \iff M = L', L = M^{-4}M' = M'L'^{-4}$. then

$$\begin{aligned}
A_K(L,M) &= A_K((M')(L')^{-4}, (L')) \\
&= -(L')^4 + (1 - (L')^{-2} - 2(L')^4 - (L')^6 + (L')^8)((M')(L')^{-4}) - (L')^4((M')(L')^{-4})^2 \\
&= -(L')^4 + (1 - (L')^{-2} - 2(L')^4 - (L')^6 + (L')^8)((M')(L')^{-4}) - (M')^2(L')^{-4} \\
&= -(L')^8 + (1 - (L')^{-2} - 2(L')^4 - (L')^6 + (L')^8)(M') - (M')^2
\end{aligned}$$

Changing an edge e of slope $-\frac{p}{q}$ to the horizontal edge at the start of the lecture: we want to work out a change of basis. All points of e are points on a straight line of slope $-\frac{p}{q}$.

$$\frac{n_j - n_i}{m_j - m_i} = \frac{-p}{q} \implies q(n_j - n_i) = -p(m_j - m_i)$$

Therefore $qn_j + pm_j = qn_i + pm_i = d$, say, for all i,j. Set $X = M^{-p}L^q, Y = M^aL^b$ where $\begin{pmatrix} -p & a \\ q & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), (p,q) = 1$. This holds if and only if $M = X^bY^{-q}, L = X^{-a}Y^{-p}$ (using det $= 1$). $A_K(L,M) = A_K(X^{-a}Y^{-p}, X^bY^{-q})$ (after multiplying by suitable powers of X, Y we get a polynomial again. The edge appears in $A_K(L,M)$ as the terms $\sum a_i L^{m_i} M^{n_i}$. Rewriting each term of this:

$$\begin{aligned}
(X^{-a}Y^{-p})^{m_i}(X^bY^{-q})^{n_i} &= X^{-am_i}Y^{-pm_i}X^{bn_i}Y^{-qn_i} \\
&= X^{-am_i+bn_i}Y^{pm_i+qn_i} \\
&= X^{-am_i+bn_i}Y^{-d}
\end{aligned}$$

where $d = pm_i + qn_i$ from above. Multiplying by Y^d clears these terms in Y, which implies that $A_K(X,Y) = \sum a_\alpha X^\alpha + E(X,Y)$ where $E(X,0) = 0 \ \forall \ X$, so there is a slope $0/1$ edge as required.

Indeed, the a_β are constants and at least two of them are non-zero, and M divides every term in the polynomial $E(L,M)$. Then $A_K(L,0) = \sum a_\beta L^\beta$ has at least two terms and hence there is a point of the form $(r, 0) \in V(A_K)$ with $r \neq 0$. This is a point at infinity, recall that the slope of an associated surface is

$$\frac{p}{q} = -\frac{\nu(L)}{\nu(M)} = 0$$

We know that $\nu(L) = 0$ because L is bounded near $(r, 0)$. Likewise, $\nu(M) \neq 0$. Now to extend this argument to show that any boundary slope of Newt(A_K) corresponds to a ∂-slope of K, we need to make a change of basis argument.

The edge polynomials come from reading off the obvious 1-variable polynomials from the edges of the Newton polygon. For slope zero edges this is straightforward, for edges of slope p/q make the substitution $t = L^{-q}M^p$.

examples:

(1) 5_2: the edge polynomial is $1 - t$. Changing basis gives a way to associate to any edge an edge polynomial $f_e(t)$.

(2) 4_1: For the edge e, the terms in the A-polynomial are $-M^4 + LM^8$. Let $t = L^{-1}M^4$, so $M^4 = \frac{t}{L}$, and we have

$$\frac{-t}{L} + L\frac{t^2}{L^2} = \frac{-t}{L} + \frac{t^2}{L} = \frac{1}{L}(-t + t^2)$$

so $f_e(t) = -t + t^2$. More generally, if e has slope $\frac{p}{q}$, let $t = L^{-q}M^p$.

Theorem 3.9. *Let $f_e(t)$ be an edge polynomial.*

(1) *$f_e(t)$ is a product of cyclotomic polynomials.*
(2) *The corner polynomials have coefficients ± 1.*

Remark: These roots are the eigenvalues of the bounded class as we go to a point at infinity corresponding to that edge. Note also, that the number of boundary components of a surface associated to a given edge is related to the roots of the edge polynomial.

Definition 30. *A* ***corner polynomial*** *is a term $a_{\alpha\beta}L^\alpha M^\beta$ in the A-polynomial such that (α, β) is a vertex of Newt(A_K).*

example: 4_1: the corner polynomials are $LM^8, -M^4L^2, -M^4, L$.

3.4. **April 8: Ones in the Corners.**

Theorem 3.10. *Let $f_e(t)$ be an edge polynomial.*

(1) *$f_e(t)$ is a product of cyclotomic polynomials. Also, if ω is a pth root of unity that satisfies the edge polynomial then p divides the number of boundary components of any connected component of an essential surface associated to that edge.*
(2) *The corner polynomials have coefficients ± 1.*

Remark: (2) implies (1). What follows is a summary of a proof of (1).

Choose a slope α which is not a boundary slope. Extend α to a basis $\{\alpha, \beta\}$ for $\pi_1(T)$. Since changing basis doesn't affect the coeffients of A_K, we have $A_K(L, M) \in \mathbb{Z}[L, M]$ where L and M are the eigenvalues of α and β. Since α is not a boundary slope, there is a unique term CM^sL^r of highest power in L. Our goal is to show that $C = \pm 1$ which by changing basis is enough to prove the theorem. We use the following claims:

(1): There exists a prime $p \in \mathbb{Q}$ such that
  - $(*, p) \in \mathrm{Im}(\xi)$, and
  - p and C are co-prime.

(2); Suppose p is a prime, $l > 0$, and $h(t) = c_0 + \cdots + c_k p^l t^k$ is an irreducible polynomial in $\mathbb{Z}[t]$ with $c_k = \pm 1$ and p doesn't divide c_k. If b satisfies h then there exists a valuation ν on $\mathbb{Q}(b)$ such that
  - $\nu(p + 1/p) \geq 0$, and
  - $\nu(b) < 0$.

We take p to be as in claim (1) and use the following lemma:

Lemma 3.11. *Factorize the one variable polynomial $A_K(L, p)$ over $\mathbb{Z}$ as*

$$A_K(L, p) \;=\; \eta \prod h_j(L)$$

where $\eta \in \mathbb{Z}$ and each h_j is irreducible over $\mathbb{Z}$. Then C divides η.

The lemma can be proved by noticing that if C doesn't divide η then claim (2) will give us a valuation. This valuation allows us to construct a non-trivial action on a tree for which α lies in a vertex stabilizer and β does not. Hence we contradict that α is not a boundary slope.

The lemma allows us to show that C divides each coefficient of $A_K(L, M)$. Since, we have normalized A_K so that the coefficients have gcd one, then $C = \pm 1$. $\square$

3.5. **April 13: Applications of Ones in the Corners.** The first is similar to Bass' Theorem:

Theorem 3.12. *Take $M = \mathbb{H}/\Gamma$ with the usual assumptions. Assume that there exists $\gamma \in \Gamma$ such that $\mathrm{Tr}(\gamma)$ is not an algebraic integer. Then M contains a closed, essential surface.*

In particular, if M is non-Haken, then $\mathrm{Tr}(\gamma)$ is an algebraic integer for every $\gamma \in \Gamma$.

Let K be a hyperbolic knot in S^3 and $\alpha \in \pi_1(T)$ a slope which is not a boundary slope. We denote the resulting closed manifold after Dehn surgery along α by $M(\alpha)$.

Theorem 3.13. *Assume $\widehat{\rho}_\alpha : \pi_1\big(M(\alpha)\big) \to \mathrm{SL}_2(\mathbb{C})$ is an irreducible representation which is non-trivial on $\pi_1(T)$. Let ξ_α be the eigenvalue of the core curve of α-Dehn surgery. Then ξ_α is a unit in the ring of algebraic integers.*

If $\alpha = m^p l^q$ then choose r and s so that

$$\det \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1.$$

Then the core curve is represented by $\beta = m^r l^s$. We introduce the parameter T where $M = T^{-q}$ and $L = T^p$. Consider the one variable polynomial $A_K(T^{-q}, T^p)$ and normalize to get the polynomial $f(T) \in \mathbb{Z}[T]$. Because α is not a boundary slope, the coefficients of $f(T)$ are the same as those of $A_K(L, M)$. By definition, $\xi_\alpha = M^r L^s$. After substituting T for M and L we see that $\xi_\alpha = T^{\pm 1}$. Thus, the minimum polynomial, $g(T)$, for ξ_α divides $f(T)$. "Ones in the corners" and Gauss' Lemma imply that $g(T)$ has leading and constant coeffients ± 1. Therefore ξ_α is an algebraic unit. $\square$

The following theorem is proved again by considering the polynomials $f(T)$ and $g(T)$. First, some definitions.

Definition 31. *If Γ is Kleinian with finite co-volume, then the* ***trace field*** *for Γ is the number field*

$$\mathbb{Q}\Big(\mathrm{Tr}(\gamma) \,\Big|\, \gamma \in \Gamma\Big).$$

also,

Definition 32. *The* ***invariant trace field*** *for Γ is the subfield of the trace field given by*

$$k\Gamma \;=\; \mathbb{Q}\Big(\big(\mathrm{Tr}(\gamma)\big)^2 \,\Big|\, \gamma \in \Gamma\Big).$$

Theorem 3.14. *Let p/q be a hyperbolic Dehn filling and $\xi_{p/q}$ the eigenvalue of the core curve. Then*

$$\Big[\mathbb{Q}(\xi_{p/q}) : \mathbb{Q}\Big] \to \infty \quad \text{as} \quad |p| + |q| \to \infty.$$

Corollary 3.15. *Let $\Gamma_{p/q} = \pi_1\big(M(p/q)\big)$, then*

$$\big[k\Gamma_{p/q} : \mathbb{Q}\big] \to \infty \quad \text{as} \quad |p| + |q| \to \infty.$$

The proof uses the following definition. First let $p \in \mathbb{Z}[x]$.

$$p(x) \;=\; a_n x^n + \cdots + a_0 \qquad \text{where } a_n \neq 0$$

Definition 33. *The L^1-**norm** of p is*

$$L(p) \; := \; \sum |a_i|.$$

*The **Mahler measure** of p is*

$$M(p) \; := \; |a_n| \prod_{p(\theta)=0} \max(1, |\theta|).$$

The properties

(1) $M(p) \leq L(p)$, and
(2) $M(p_1 \cdot p_2) = M(p_1) \cdot M(p_2)$.

are easily established.

Proof: Corresponding to each p/q we get a polynomial $f(T)$ as before. Since the coefficients of f are the same as those of A_K, we have that $L(f)$ is bounded. By property (1), $M(f)$ is bounded. If we assume that $\left[\mathbb{Q}(\xi_{p/q}) : \mathbb{Q}\right]$ is bounded, it can be shown that there exist only finitely many possible polynomials $g(T)$. Then there are only finitely many possible values for ξ. This contradicts that $\chi_\rho(\alpha)$ is non-constant. $\square$

3.6. **April 15: Cyclic Surgery and Boundary Slopes.** Let M be a compact, orientable, irreducible 3-manifold with ∂M and incompressible torus. If α is a slope such that $\pi_1\big(M(\alpha)\big)$ is cyclic, then α is called a cyclic slope.

Theorem 3.16 (Dunfield)**.** *Assume that $K \subset \mathrm{S}^3$ is a small, hyperbolic knot and β a non-trivial, cyclic slope. Then there exists a non-integral ∂-slope, r_γ such that*

$$r_\gamma \ = \ \big(r_\beta - 1, r_\beta + 1\big).$$

Remarks:

- By the Cyclic Surgery Theorem, $r_\beta \in \mathbb{Z}$.
- If K is the $(-2, 3, 7)$-pretzel knot, K has 18 and 19 as cyclic slopes and the non-integral slope 37/2 in the required interval.
- If $K(n)$ is the knot obtained by the below surgery description, let $M_n = \mathrm{S}^3 \backslash K(n)$. $M(9n)$ is a lens space. Also, the 2-fold branched cover of $K(2n)$ is $-1/n$-Dehn Surgery on the figure eight knot. By Gordon-Litherland, $K(2n)$ is small. Hence, we see a family of examples to which we may apply Dunfield's theorem.

The theorem follows from Gabai's Property R results, the following theorem from CGLS, and the following proposition.

Theorem 3.17 (CGLS, Theorem 2.0.3). *Let r be a boundary slope. Then one of the following holds.*

- *$M(r)$ is Haken. (non-cyclic fundamental group)*
- *$M(r)$ is the connect sum of two lens spaces. (non-cyclic fundamental group)*
- *M contains a closed, essential surface. (K would not be small)*
- *r is a slope of a planar surface and M fibers over the circle with this surface as a fiber. (Gabai's property R rules this out since 0-surgery on K is irreducible (not $\mathrm{S}^1 \times \mathrm{S}^2$.))*

Let β be a non-trivial cyclic slope. Then by CGLS, β is integral and $\{m, \beta\}$ is a basis for $\pi_1(T)$. The below proposition is with respect to this basis.

Proposition 3.18. *There exists a boundary slope s_γ such that $|s_\gamma| < 1$.*

If $\gamma \in \pi_1(M)$ we define the function $f_\gamma : X\big(\pi_1(M)\big) \to \mathbb{C}$ by

$$f_\gamma \;=\; \big(\mathrm{Tr}(\rho(\gamma))\big)^2 - 4.$$

Lemma 3.19. *Let M be a finite volume, hyperbolic 3-manifold with a single cusp. Assume that β is a non-trivial cyclic slope and $\langle\, \mu, \beta \,\rangle = \pi_1(T)$, then either*

- *there exists a boundary slope s_γ with $|s_\gamma| < 1$, or*
- *$\frac{f_\mu}{f_\beta}$ is constant on X_0.*

We can rule out the second possibility with a "ones in the corners"-type argument. An outline follows. To avoid some non-trivial difficulties, we will assume that A_0 is defined over $\mathbb{Q}$.

Assume that $\frac{f_\mu}{f_\beta} = C'$ on X_0. Since f_μ is non-constant we see that C' cannot be zero. After a bit of algebra, we have

$$\left(\frac{M - M^{-1}}{B - B^{-1}}\right)^2 \;=\; C'$$

on $V_0 = V\Big(A_0(B, M)\Big)$. Since V_0 is irreducible, we can make a choice of square root, say C, of C'. So that

$$F(B, M) \;=\; M^2 B - B - CMB^2 + CM$$

is zero on V_0. Now using our assumption that A_0 is defined over $\mathbb{Q}$, we can show that $C \in \mathbb{Q}$. Also, if $C = \pm 1$ then F factors into either

$$(BM + 1)(B - M) \qquad \text{or} \qquad (BM - 1)(B + M)$$

but none of these factors can be constant on V_0. Hence, $C \in \mathbb{Q} - \{\pm 1\}$. As such, it can be shown that F is irreducible. Therefore $A_0 = F$.

Now, we use the fact that the edge polynomials of $\mathrm{Newt}(F)$ must divide the edge polynomials of A_K. We have an edge associated to the terms $BM^2 + CM$. Making the substitution $T = BM$ we have the edge polynomial $T - C$. Therefore, $T - C$ divides a cyclotomic polynomial. Since, C is a root of unity and a rational number we arrive at the contradiction $C = \pm 1$.

Therefore, we have ruled out case two of Lemma 3.19, and so there exists a boundary slope s_γ, with $|s_\gamma| < 1$, as stated in Proposition 3.18. $\square$

3.7. **April 20: Proving Lemma 3.19.** Recall the lemma.

Let M be a finite volume, hyperbolic 3-manifold with a single cusp. Assume that β is a non-trivial cyclic slope and $\langle \mu, \beta \rangle = \pi_1(T)$, then either

- there exists a boundary slope s_γ with $|s_\gamma| < 1$, or
- $\frac{f_\mu}{f_\beta}$ is constant on X_0.

Proof: First note that if β is a boundary slope, then the first conclusion is satisfied since $|s_\beta| = 0$. Thus we assume that β is not a boundary slope and $g = \frac{f_\mu}{f_\beta}$ is non-constant on $\widetilde{X}_0$. Since g is non-constant, it has zeros and poles. If $x \in \widetilde{X}_0$, define

$$Z_x(h) = \begin{cases} \text{order of the zero at } x \text{ for } h, \text{or} \\ 0 \text{ if } h(x) \neq 0 \end{cases}$$

and

$$\Pi_x(h) = \begin{cases} \text{order of the pole at } x \text{ for } h, \text{or} \\ 0 \text{ if } h(x) \in \mathbb{C} \end{cases}$$

Note that if x is a point at infinity and ν_x is the associated valuation, then

$$\nu_x(h) = \text{ord}_x(h) = \begin{cases} Z_x(h) & \text{if } h(x) = 0 \\ 0 & \text{if } h(x) \in \mathbb{C} - \{0\} \\ \Pi_x(h) & \text{if } h(x) = \infty \end{cases}$$

Take $y \in \widetilde{X}_0$ so that g has a pole there. So we have two cases.

Case 1: $f_\beta(y) = 0$.

Claim 1: $f_\mu(y) = 0$ and $Z_x(f_\mu) \geq Z_x(f_\beta)$.

This leads immediately to a contradiction since g has a pole at y.

Case 2: f_μ has a pole at y, and therefore y is a point at infinity. Hence we have an essential surface, Σ, with non-empty boundary. Let γ be the boundary slope in $\{\mu, \beta\}$ coordinates.

Claim 2: $|s_\gamma| = \frac{\Pi_y(f_\beta)}{\Pi_y(f_\mu)}$.

Since g has a pole at y, $\Pi_y(f_\mu) > \Pi_y(f_\beta)$. Therefore,

$$1 > \frac{\Pi_y(f_\beta)}{\Pi_y(f_\mu)} = |s_\gamma|. \quad \square$$

Claim 1 follows from Proposition 1.1.3 in CGLS since β is a cyclic slope which is not a strict boundary slope.

Proposition 3.20 (CGLS 1.1.3)**.** *Let $\alpha \in \pi_1(T)$ be a slope which is not a strict boundary slope and so that $\pi_1\big(M(\alpha)\big)$ is cyclic. Then for any $x \in \widetilde{X}_0$ we have*

$$Z_x(f_\alpha) \leq Z_x(f_\delta)$$

for any non-trivial $\delta \in \pi_1(T)$.

Outline: Assume that there exists such a δ and x so that $Z_x(f_\delta) < Z_x(f_\alpha)$.

Case 1: Assume that x is a point at infinity. Then x detects a CES, Σ, since $f_\alpha(x) = 0$ and α is not a strict boundary slope. CGLS shows that Σ remains incompressible in $M(\alpha)$. This contradicts that α is a cyclic slope.

Case 2: Assume that x is not a point at infinity. Since $f_\alpha(x) = 0$, if $\rho \in t^{-1}(x) \cap R_0$ then $\rho(\alpha)$ is parabolic. From CGLS, every such representation maps α to $\pm I$ in $\mathrm{SL}_2(\mathbb{C})$. Now, as in the proof for the Smith Conjecture, there exists $\rho \in ({}^{-1}t)(x) \cap R_0$ with non-cyclic image. We then get a non-cyclic representation $\pi_1\big(M(\alpha)\big) \to \mathrm{SL}_2(\mathbb{C})$. This contradicts that α is a cyclic slope. $\square$

Lastly, we show Claim 2.

$$|s_\gamma| \;=\; \frac{\Pi_y(f_\beta)}{\Pi_y(f_\mu)}$$

Let $\alpha = \mu^r\beta^s$ and normalize so that

$$\rho(\mu) \;=\; \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix} \qquad \text{and} \qquad \rho(\beta) \;=\; \begin{pmatrix} B & t \\ 0 & B^{-1} \end{pmatrix},$$

then

$$\rho(\alpha) \;=\; \begin{pmatrix} M^rB^s & * \\ * & \frac{1}{M^rB^s} \end{pmatrix}.$$

Recall that if p/q is a boundary slope detected at y then

$$\frac{p}{q} \;=\; -\frac{\nu_y(B)}{\nu_y(M)}.$$

Also, recall that if (F, v) is a valued field and $x \in F^\times - \{\pm 1\}$ then $|v(x)| = -\min\big(0, v(x - x^{-1})\big)$.

Using the valuation fact, we can show that $\Pi_y(f_\alpha) = 2|\nu_y(M^rB^s)|$.

Now using the boundary slope fact, we have

$$\frac{\Pi_y(f_\beta)}{\Pi_y(f_\mu)} \;=\; \frac{\big|\nu_y(B)\big|}{\big|\nu_y(M)\big|} \;=\; |s_\gamma|. \quad \square$$