

DERHAM COHOMOLOGY OF THE RECTANGULAR TORUS

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ABSTRACT. For the special case of a rectangular flat torus, we present and prove DeRham's Theorem, which says that cohomology is given by closed differential forms modulo exact forms.

1. HOMOLOGY AND COHOMOLOGY OF THE TORUS

Homology and cohomology use combinatorial algebra to capture topological information, to distinguish a torus from a sphere for example. DeRham's Theorem gives an equivalent analytic definition of cohomology in terms of differential forms. We will illustrate the theory with the example of a flat rectangular two-torus \mathbb{T}^2 .

Homology is defined roughly as cycles (closed curves) modulo boundaries of regions. Since every closed curve on the sphere is a boundary, the homology of the sphere is trivial. On the other hand, on the torus \mathbb{T}^2 , there are many types of closed curves which are not boundaries and because of this the homology is nontrivial. Cohomology is the dualspace. DeRham's Theorem gives an alternative definition of cohomology in terms of differential forms.

This paper began as an undergraduate colloquium talk, required of every Williams College Mathematics and Statistics Major. I would like to thank Professor Frank Morgan for advising the talk and this write-up.

1.1. **Definitions.** Consider an $L \times W$ rectangular torus $\mathbb{T}^2 = (\mathbb{R} \bmod L) \times (\mathbb{R} \bmod W)$ with coordinates (x, y) . *0-chains* are sets of points with positive or negative orientation, *1-chains* are oriented immersed curves, and *2-chains* are oriented immersed surfaces. The *boundary* of a k -chain is a $(k - 1)$ -chain. Specifically, the boundary of a 1-chain consists of the oriented sum of its endpoints, and the boundary of a 2-chain consists of its bounding

Date: April 23, 2002.

curves. A *cycle* is a closed chain, where *closed* means without boundary. A boundary itself has no boundary.

The *homology* is the set of all cycles modulo boundaries. In particular, the first homology is a set of equivalence classes of 1-cycles, where two 1-cycles are homologically equivalent if and only if their difference bounds a piece of surface, as the difference of two meridians bounds a meridional strip. Two single points with the same orientation are homologically equivalent in the zeroth homology because their difference bounds a curve (since \mathbb{T}^2 is a connected space). The cohomology is the dual to the homology, the set of all linear functionals on the homology.

A *differential 0-form* on \mathbb{T}^2 is just a real-valued smooth function, $f(x, y)$ on \mathbb{T}^2 . Similarly, we have 1-forms $\omega = M(x, y)dx + N(x, y)dy$ and 2-forms $\varphi = F(x, y)dxdy$. Given a 0-form f , the associated 1-form

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

is called the *exterior derivative* of f . The exterior derivative of the 1-form $\omega = M(x, y)dx + N(x, y)dy$ is the 2-form $d\omega = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$. Further, ω is a *closed* differential form if $d\omega = 0$, and ω is an *exact* differential form if $\omega = d\tau$ for some differential form τ . An important fact is that $d^2 = d \circ d = 0$, and thus the set of exact forms is a subset of the set of closed forms. DeRham's Theorem 3.1 will say that the cohomology is given by the closed forms modulo exact forms. Frenkel [Fre] and Bott [Bot82] provide good references on differential forms.

2. STOKES'S THEOREM AND VANISHING PATH INTEGRALS

Next we state and prove the primary lemmas that we will use to prove DeRham's theorem. The first lemma below is Stokes's Theorem.

Lemma 2.1. *Let R be an oriented region on \mathbb{T}^2 and let φ be a differential form on R . Then*

$$\int_{\partial R} \varphi = \int_R d\varphi$$

For a proof of Lemma 2.1, see Marsden and Tromba [MT81].

Lemma 2.2. *If the integral $\oint \varphi$ of a 1-form φ along any closed path vanishes, then φ is exact.*

Proof. Since $\oint \varphi = 0$, $\int_a^b \varphi$ is path independent, and thus we can define a function by fixing $a \in \mathbb{T}^2$ and defining $f(b)$ for $b \in \mathbb{T}^2$ by

$$f(b) = f(a) + \int_a^b \varphi.$$

Then $df = \varphi$ by the fundamental theorem of calculus for path integrals, and thus φ is exact as claimed. □

3. DERHAM'S THEOREM

Here we state and prove the main result that this paper explores, namely DeRham's theorem for the case of the first cohomology of a rectangular torus.

Theorem 3.1. *The space of closed differential 1-forms modulo exact 1-forms on the rectangular torus \mathbb{T}^2 is dual to the first homology of the rectangular torus.*

In other words, cohomology is given by closed forms modulo exact forms.

Proof. Define the action of $[\varphi]$ on $[C]$ as $\int_C \varphi$, which is clearly linear in φ and C . To show that the action is well defined, first we show that exact forms $\varphi = df$ get mapped to zero. Indeed, for any closed curve C ,

$$\int_C \varphi = \int_C df = \int_{\partial C} f = 0$$

by Stokes's Theorem and the fact that closed curves have no boundary. Second we show that integrals over boundaries ∂R vanish. Indeed, for any closed form φ ,

$$\int_{\partial R} \varphi = \int_R d\varphi = 0$$

by Stokes's Theorem again and the fact that $d\varphi = 0$.

It remains to show that our map from $[\varphi]$ to $\int_C \varphi$ is injective and surjective. In order to prove injectivity, we must show that if $\oint \varphi = 0$ for any closed path, then φ is an exact form. This follows immediately from lemma 2.2.

Finally, to prove surjectivity, we show that the homology (and hence the dual space or cohomology) has dimension at most two. This will suffice because we know that there are at least two linearly independent closed forms, namely dx and dy .

It suffices to show that if $\int_C dx = \int_C dy = 0$, then C is a boundary. This hypothesis implies that C lifts to a *closed* curve in the plane, where it bounds some region R . Now C bounds the projection of R in \mathbb{T}^2 , so that C is a boundary, as desired.

□

4. CONCLUSION

We have shown that in the special case of the rectangular two-torus \mathbb{T}^2 , the cohomology is given by closed forms modulo exact forms. This is an exciting result because it is an analytic reformulation of an object which is algebraic in nature. Surprisingly, this relationship holds in general, as we state as Theorem 4.1 without proof.

Theorem 4.1. *The space of closed differential k -forms modulo exact k -forms on a smooth manifold M is dual to the k^{th} homology of M .*

We refer the interested reader to [ST67] which includes a proof of this theorem.

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