Name: answers
Class time:

M305G - Practice Final Exam - December 2008

Carefully read each question, and be sure to show your work.

1. (10 points) Consider the function $R(x) = x - \frac{1}{x}$. $= \frac{x^2}{x} - \frac{1}{x} = \frac{x^2-1}{x}$

   (a) If we write $R(x) = \frac{p(x)}{q(x)}$, then what is the polynomial $p(x)$? And what is $q(x)$?

   $p(x) = x^2 - 1$ $\quad q(x) = x$

   (b) What is the domain of $R(x)$? What is the y-intercept of the graph of $R(x)$?

   domain of $R(x) = \{x \mid x \neq 0\}$

   there is no y-intercept

   (c) What are the x-intercepts of the graph of $R(x)$?

   x-int's = zeros of $p(x)$: $x = \pm 1$

   (d) What are the vertical asymptotes of the graph of $R(x)$?

   VA's = zeros of $q(x)$: $x = 0$

   (e) Does $R(x)$ have a horizontal or oblique asymptote? If so, what is it?

   OA : $y = x$

2. Let $f(x) = 2 + \log_3(x-1)$. Note: f is a one-to-one function.

(a) What is $f(2)$? And what is $f(4)$?

$f(2) = 2 + \log_3(2-1) = 2 + \log_3 1 = 2$

$f(4) = 2 + \log_3(4-1) = 2 + \log_3 3 = 3$

(b) What is $f^{-1}(x)$?

$x = 2 + \log_3(y-1)$

$x - 2 = \log_3(y-1) \rightarrow 3^{x-2} = y - 1$

$\boxed{f^{-1}(x) = 1 + 3^{x-2}}$

(c) What is the domain of f? And what is the range of f?

domain of $f = \{x \mid x > 1\}$

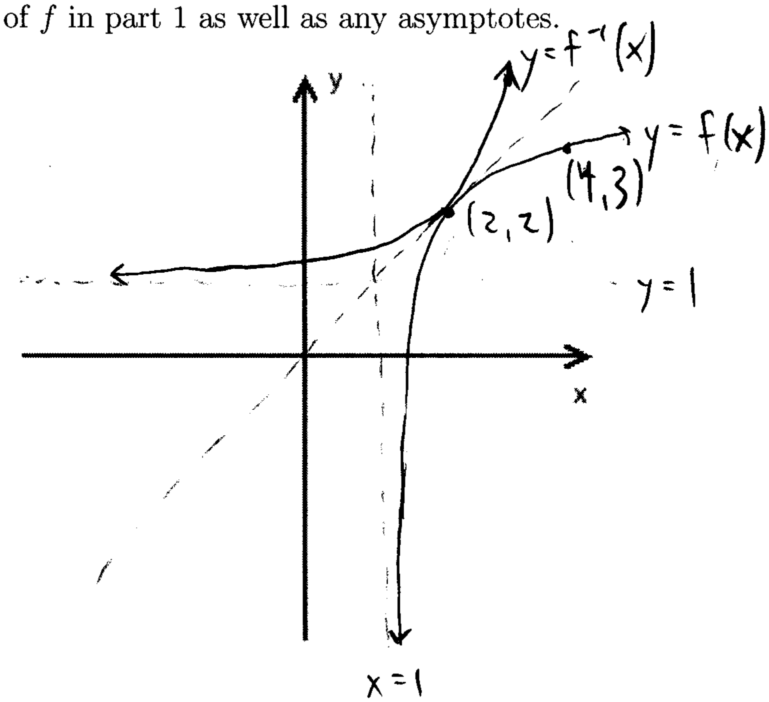
range of f = domain of $f^{-1} = \mathbb{R}$

(d) Using properties of logarithms, write $f(x)$ as a single logarithm. (Hint: What is $\log_3 9$?)

$f(x) = 2 + \log_3(x-1) = \log_3 9 + \log_3(x-1)$

$= \boxed{\log_3(9x-9)}$

(e) Graph f and f^{-1} on the same xy-plane below. Clearly label both points you found to be on the graph of f in part 1 as well as any asymptotes.



3. (10 points) This problem involves piecewise-defined functions AND composite functions.

Let

$$f(x) = \begin{cases} 4x+8 & x \geq -2 \\ x+2 & x < -2 \end{cases}$$

and

$$g(x) = 2x + 1$$

(a) What is $(f \circ g)(1)$?

$$f(g(1)) = f(3)$$
$$= 4(3) + 8 = \boxed{20}$$

(b) What is $(g \circ f)(-3)$?

$$g(f(-3)) = g(-3+2)$$
$$= g(-1) = 2(-1) + 1 = \boxed{-1}$$

(c) Is f one-to-one? Why or why not?

YES because its graph passes the HLT

(d) Consider the two cases $x \geq -2$ and $x < -2$ to solve

$$\frac{f(x)}{g(x)} \leq 0$$

Case 1: $x \geq -2$

$$\frac{4x+8}{2x+1} \leq 0$$

$-$ $+$

-2 $-\frac{1}{2}$

$\{x \mid -2 \leq x < -\frac{1}{2}\}$

Case 2: $x < -2$

$$\frac{x+2}{2x+1} \leq 0$$

$+$

-2

no solutions when $x < -2$

$$\boxed{\text{So the solution set is } \{x \mid -2 \leq x < -\tfrac{1}{2}\}}$$

4. (10 points) Let $g(x) = \frac{1}{2}|x+1| - 2$.

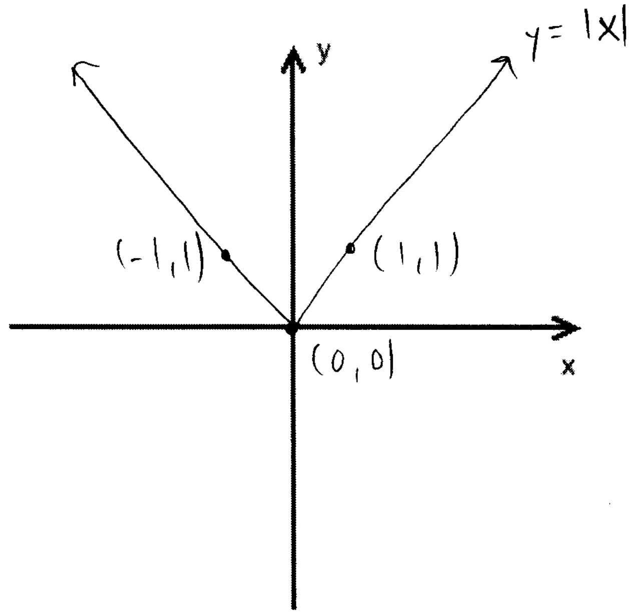
(a) Write $g(x)$ as a piecewise-defined function:

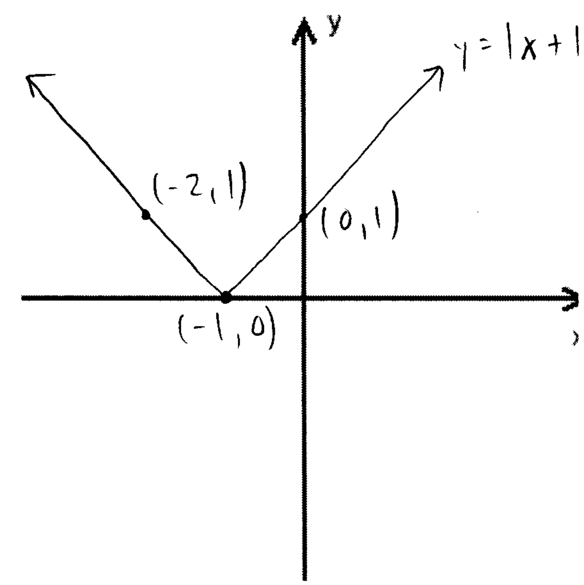
$x+1 \geq 0 \Rightarrow \frac{1}{2}x + \frac{1}{2} - 2$

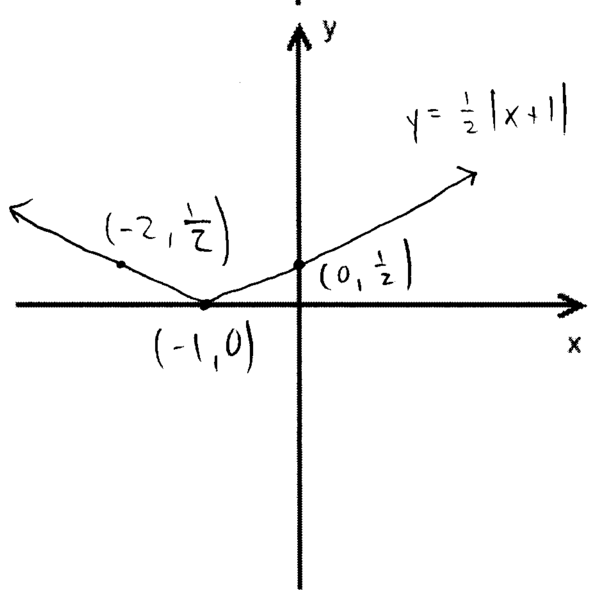
$x+1 < 0 \Rightarrow \frac{1}{2}(-x-1) - 2$

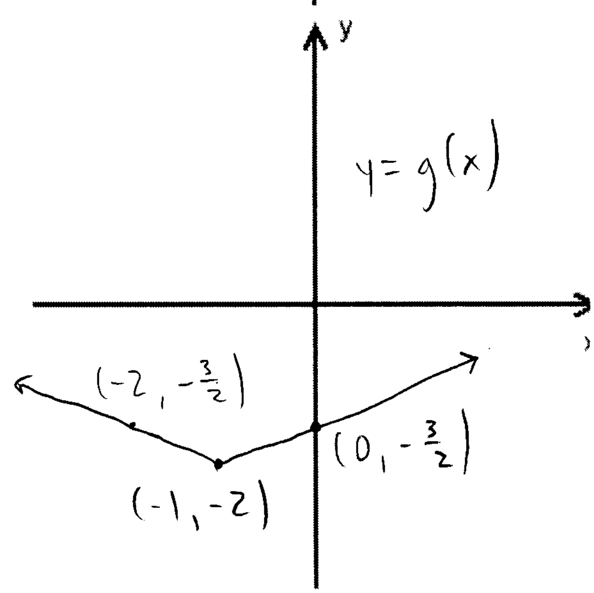
$$g(x) = \begin{cases} \frac{x}{2} - \frac{3}{2} & x \geq -1 \\ -\frac{x}{2} - \frac{5}{2} & x < -1 \end{cases}$$

(b) Starting with the graph of the basic function from your library of functions and showing all stages, graph the function $g(x)$ using the techniques of shifting, compressing, stretching, and/or reflecting. Label at least three points on each graph.









5. (10 points) Recall that a general quadratic function $f(x) = ax^2 + bx + c$ may be rewritten $f(x) = a(x-h)^2 + k$ for appropriate h and k. For this problem, let $g(x)$ be the quadratic function

$$g(x) = 2x^2 - x - 3. = (2x-3)(x+1)$$

(a) Find h and k for $g(x)$. Without drawing the graph of $g(x)$, determine its vertex and axis of symmetry.

$a = 2 \quad b = -1 \quad c = -3$

$h = -\frac{b}{2a} = \boxed{\frac{1}{4}}$

$\boxed{\text{vertex} = (h,k) = \left(\frac{1}{4}, \frac{-25}{8}\right)}$

$k = g(h) = g\left(\frac{1}{4}\right) = 2\left(\frac{1}{16}\right) - \frac{1}{4} - 3 = \frac{1}{8} - \frac{1}{4} - 3$

$\boxed{\text{axis} : x = \frac{1}{4}}$

$= \boxed{-\frac{25}{8}}$

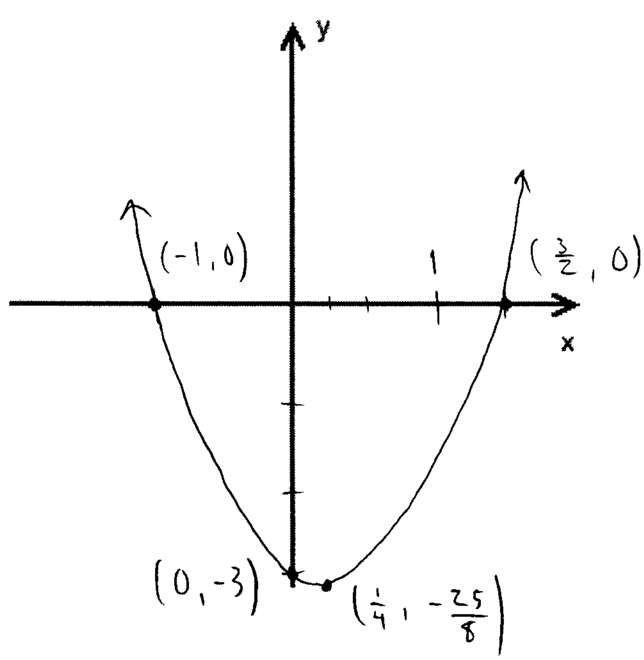
(b) Find the y-intercept of $g(x)$.

y-int : $\boxed{y = g(0) = -3}$

(c) What are the x-intercepts of $g(x)$?

zeros of g : $\boxed{x = \frac{3}{2}, \quad x = -1}$

(d) Graph $g(x)$ below. Label all points that you found in parts (a) through (c).



6. (10 points) In this problem, you will create and graph a polynomial function.

(a) Form a polynomial $p(x)$ having ALL OF THE FOLLOWING as real zeros:

- $x = -2$ of multiplicity 2
- $x = 0$ of multiplicity 1
- $x = 3$ of multiplicity 2

(Note: there is more than one right answer for this problem.)

$p(x) = x(x+2)^2(x-3)^2$

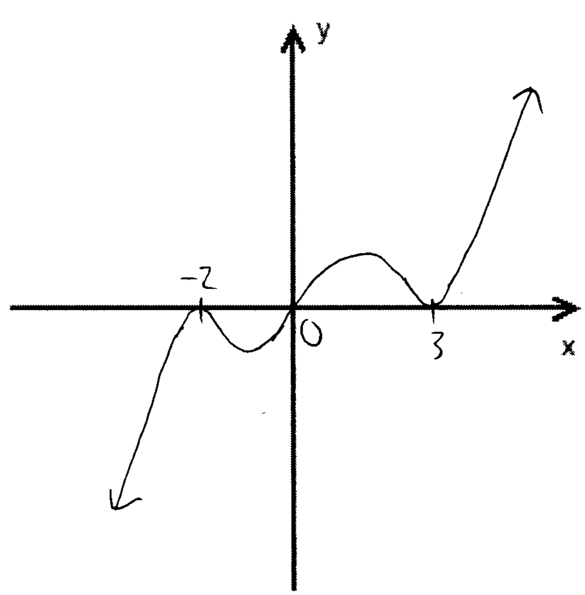
(b) What is the degree of your polynomial?

5

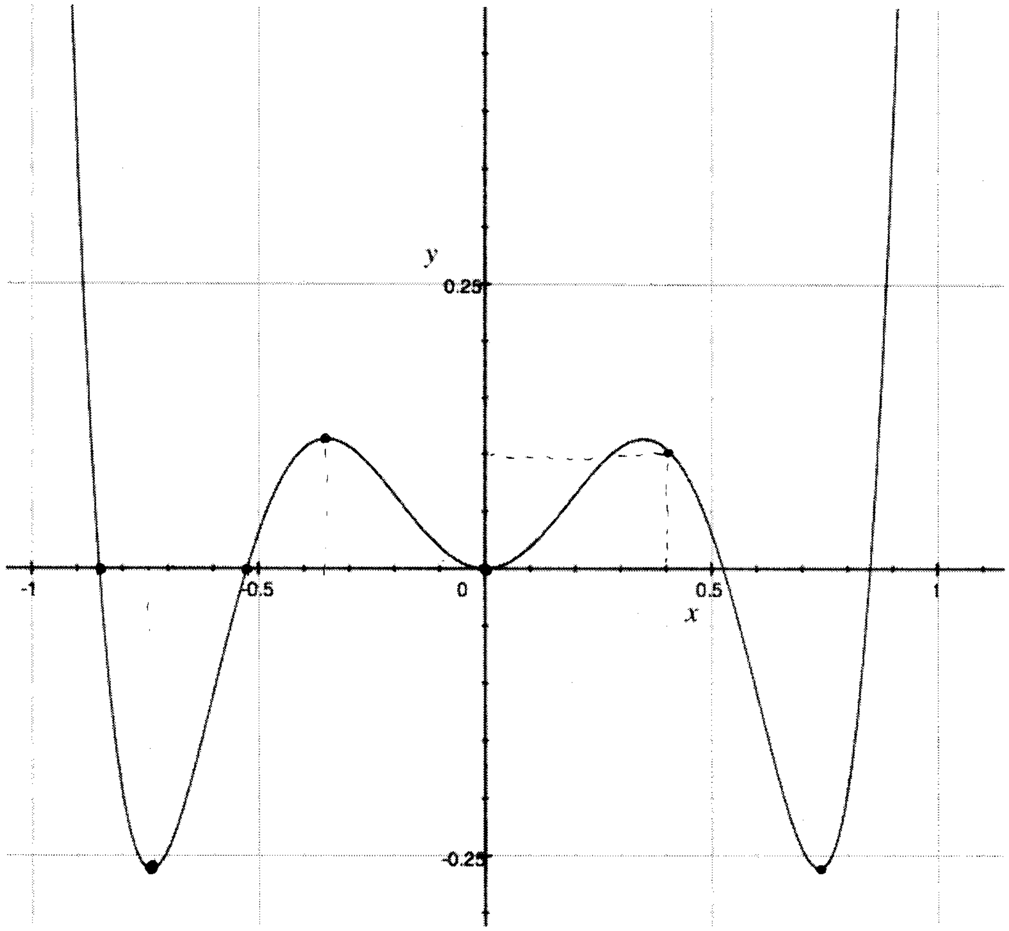
(c) What is the y-intercept of the graph of $p(x)$?

0

(d) Sketch the graph of your polynomial. Label all intercepts.



7. (10 points) Consider the graph of a degree 6 polynomial $f(x)$, which happens to be an even function:



For each of the following questions, estimate the answer using the information from the graph.

(a) What is $f(0.4)$?

≈ 0.10

(b) Does the graph of f have any symmetry? If so, what kind does it have?

yes, symmetry across the y-axis

(c) What are the x-coordinates of the local minima of this graph? What are the values of the function at each of them?

x-coords: $-0.75,\ 0,\ 0.75$

values: $-0.27,\ 0,\ -0.27$

(d) On what intervals is this function increasing?

$(-0.75, -0.36),\ (0, 0.36),\ (0.75, \infty)$

(e) On what interval(s) is $f(x) \leq 0$?

$[-0.85, -0.52],\ [0.52, 0.85]$ (and at $x = 0$)

8. (10 points) Solve the following equations:

(a)

$$e^{4x} = e^{x^2} \cdot \frac{1}{e^4} = e^{x^2-4}$$

$$\Rightarrow 4x = x^2 - 4$$

$$0 = x^2 - 4x - 4 = (x-2)^2$$

$$\boxed{x = 2}$$

(b)

$$\sec\left(\sin^{-1}\frac{2\sqrt{5}}{5}\right)$$

$$\theta \Rightarrow \sin\theta = \frac{2}{\sqrt{5}} \qquad y = 2, \quad r = \sqrt{5} \Rightarrow x = 1$$

$$\sec\theta = \frac{r}{x} = \boxed{\sqrt{5}}$$

(c)

$$\log_2(x-1) + 2\log_4(x-4) = 2$$

(Hint: use the change of base formula to rewrite $\log_4(x-4)$ in terms of $\log_2$.)

$$\log_4(x-4) = \frac{\log_2(x-4)}{\log_2 4} = \frac{\log_2(x-4)}{2}$$

$$\log_2(x-1) + \log_2(x-4) = 2$$

$$\log_2[(x-1)(x-4)] = 2 \Rightarrow x^2 - 5x + 4 = 4$$

$$x(x-5) = 0$$

$$\cancel{x=0} \quad x = 5$$

(b/c 0 is not in the domain of $\log_2(x-1)$)

(d)

$$3 \cdot 4^x - 4 \cdot 2^x = -1$$

$$3 \cdot (2^x)^2 - 4 \cdot 2^x + 1 = 0$$

$$3u^2 - 4u + 1 = 0$$

$$(3u-1)(u-1) = 0$$

$$u = \frac{1}{3} \qquad u = 1$$

$$2^x = \frac{1}{3} \qquad 2^x = 1 \Rightarrow \boxed{x = 0}$$

$$\boxed{x = \log_2 \frac{1}{3}}$$

9. (10 points) Establish each identity.

(a) $\frac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta} = 1 - \tan\alpha\tan\beta$

$$LHS = \frac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta} = \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta}$$

$$= \frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}$$

$$= 1 - \tan\alpha\tan\beta = RHS \checkmark$$

(b) $\sec(2\theta) = \frac{\sec^2\theta}{2-\sec^2\theta}$

$$LHS = \sec(2\theta)$$

$$= \frac{1}{\cos(2\theta)}$$

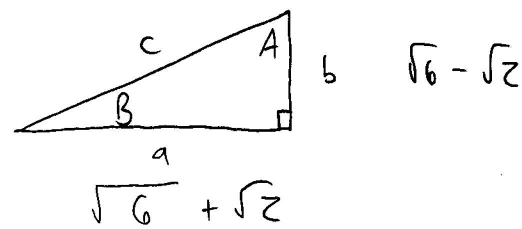
$$= \frac{1}{\cos^2\theta - \sin^2\theta} \cdot \frac{\sec^2\theta}{\sec^2\theta}$$

$$= \frac{\sec^2\theta}{1-\tan^2\theta}$$ recall: $\tan^2\theta = \sec^2\theta - 1$

$$= \frac{\sec^2\theta}{2-\sec^2\theta} = RHS \checkmark$$

10. Suppose you have a right triangle with two non-hypotenuse sides a and b of lengths $\sqrt{6}+\sqrt{2}$ and $\sqrt{6}-\sqrt{2}$ respectively, and let A and B denote the angles opposite sides a and b.

(a) What is the length of the hypotenuse, c?



$$(\sqrt{6}+\sqrt{2})^2 = 6 + 2\sqrt{12} + 2$$
$$+\ (\sqrt{6}-\sqrt{2})^2 = 6 - 2\sqrt{12} + 2$$
$$c^2 = 16 \Rightarrow \boxed{c = 4}$$

(b) What is the measure of angle A?

$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

recall: $\sin 75^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

so $\boxed{A = 75^\circ}$