5. (10 points) Evaluating trigonometric expressions. Find the exact value of each of the following expressions.

(a). $\sin\frac{\pi}{4}$

$\boxed{\frac{\sqrt{2}}{2}}$

(b). $\sec^{-1}\frac{2\sqrt{3}}{3} = \theta \Rightarrow \sec\theta = \frac{2\sqrt{3}}{3} = \frac{2}{\sqrt{3}}$

$0 \le \theta \le \pi$

$\cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \boxed{\theta = \frac{\pi}{6}}$

(c). $\tan\left(\sin^{-1}\frac{1}{3}\right)$

$\sin^{-1}\frac{1}{3} = \theta \quad , \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\sin\theta = \frac{1}{3} \Rightarrow y = 1 \quad , \quad r = 3 \quad , \quad x = 2\sqrt{2}$

$\tan\theta = \frac{y}{x} = \frac{1}{2\sqrt{2}} = \boxed{\frac{\sqrt{2}}{4}}$

(d). $\sin\left(\sin^{-1}\frac{1}{2} + \cos^{-1}0\right)$ [Hint: see problem 8 for the sum and difference formulas.]

$\sin^{-1}\frac{1}{2} = \alpha$

$\sin\alpha = \frac{1}{2}$

$\Rightarrow \cos\alpha = \frac{\sqrt{3}}{2}$

$\cos^{-1}0 = \beta$

$\cos\beta = 0$

$\Rightarrow \sin\beta = 1$

$\alpha = \frac{\pi}{6}$

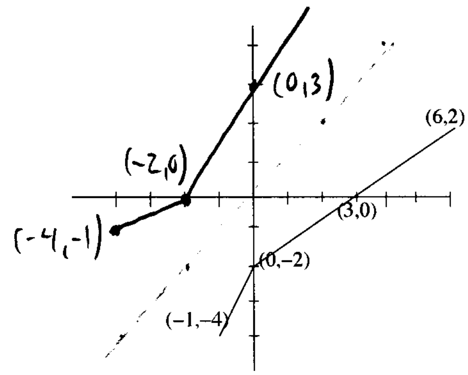
$\beta = \frac{\pi}{2}$

$\Rightarrow \alpha + \beta = \frac{2\pi}{3}$

$\boxed{\text{ans} = \frac{\sqrt{3}}{2}}$

(e). $\frac{1}{\csc^2\left(\frac{37\pi}{18}\right)} + \cos^2\left(-\frac{\pi}{18}\right)$

6. (10 points) Functions and their graphs. Consider the graph given below, of some function $f(x)$.




(a). How do you know that this is the graph of a function?

It passes the VLT

(b). On what intervals is the function $f(x)$ decreasing?

(c). On the picture of the graph, sketch a graph of the inverse of this function.

(d). Is the inverse of $f(x)$ a function? Why or why not?

(e). $f(x)$ is a piecewise defined function, composed of two line segments. Write an equation for $f(x)$.

$$f(x) = \begin{cases} 2x - 2 & -1 \le x \le 0 \\ \frac{2}{3}x - 2 & 0 \le x \le 6 \end{cases}$$

8. (10 points) Recall the following sum and difference formulas for sine and cosine.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

(a). Find the exact value of $\sin\frac{5\pi}{12}$.

$$\frac{5\pi}{12} = \frac{3\pi}{12} + \frac{2\pi}{12}$$

$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\sin\frac{\pi}{6} = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\frac{1}{2}$$

$$= \boxed{\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)}$$

(b). Use these formulas to prove the following double angle formula for cosine:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta.$$

(c). Using the trigonometric identities, establish the identity

$$1 - \frac{\cos^2\theta}{1+\sin\theta} = \sin\theta$$

$$\text{LHS} = 1 - \frac{\cos^2\theta(1-\sin\theta)}{1-\sin^2\theta} = 1 - \frac{\cos^2\theta - \cos^2\theta\sin\theta}{\cos^2\theta}$$

$$= 1 - \frac{\cos^2\theta}{\cos^2\theta} + \frac{\cos^2\theta\sin\theta}{\cos^2\theta}$$

$$= \sin\theta = \text{RHS} \checkmark$$

9. (10 points) Logarithmic and Exponential Equations. Solve for $x$ in each of the following equations.

(a). $4^{1-2x} = 2$

$2^{2-4x} = 2$

$\Rightarrow 2 - 4x = 1$

$4x = 1 \Rightarrow \boxed{x = \frac{1}{4}}$

(b). $\log_6(x+3) + \log_6(x+4) = 1$

$\log_6((x+3)(x+4)) = 1$

$x^2 + 7x + 12 = 6$

$x^2 + 7x + 6 = 0$

$(x+6)(x+1)$ $\quad \cancel{x = -6} \quad \boxed{x = -1}$

(c). $\log(7x - 12) = 2\log x$

$\log(7x - 12) - \log x^2 = 0$

$\log\left(\frac{7x-12}{x^2}\right) = 0$

$\frac{7x-12}{x^2} = 1$

$x^2 = 7x - 12$

$x^2 - 7x + 12 = 0$ $\quad (x-3)(x-4) = 0$

$\boxed{x = 3, 4}$