DISKS IN THE HEISENBERG GROUP

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1. Outline

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    - (ii) Cannon
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    - (ii) Definition I: machines
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- (2) Lemmas
  - (a) short representatives: proof by machines (definition I)
  - (b) small disks: proof by pictures (definition II)
- (3) Heisenberg group
  - (a) $\mathcal{H} =< \alpha, \beta, \gamma | [\alpha, \beta]\gamma^{-1}, [\alpha, \gamma], [\beta, \gamma] >$
  - (b) $G =< \alpha(x, y, z) = (x+1, y, z+y), \beta(x, y, z) = (x, y+1, z), \gamma(x, y, z) = (x, y, z+1) >$ implies that $\mathcal{H} \cong H = \mathbb{R}^3/G$
  - (c) use G acting on $\mathbb{Z}^3 \in \mathbb{R}^3$ to construct Cayley graph of H in $\mathbb{R}^3$
  - (d) show slides of empty Γ, with w_1, w_2
  - (e) in general, yz-area of disk D spanning w_n has area close to n^3
  - (f) elementary spanning surfaces correspond to conjugates of relators
    - (i) area of S_i (and its conjugates) ≤ 1, so need at least n^3 of them to fill D
    - (ii) so minimal $\phi(n)$ is at least cubic, therefore not automatic by [2]
    - (iii) note: combinatorial argument implies that minimal $\phi(n)$ is cubic

2. Motivation

Much of this section is adapted from [Far92].

2.1. **History.**

In the 1910s, Dehn solved the word problem for surface groups using the geometry of $\mathbb{H}^2$. In 1984, Cannon extended this to cocompact discrete groups of hyperbolic isometries. The main thrust of Cannon's paper is that one can "see" the Cayley graphs for such groups, i.e. there is a way to take a picture of the graph in a finite ball around 0 and a finite machine that dictates how to stick copies of that finite piece together. Example: $\Gamma_{\{1\}}(\mathbb{Z})$. To formalize this notion, use **finite-state automata** (FSA).

Definition 1. *Let G be a finitely generated group with generating set S and let $A = S \cup S^{-1} = \{a_1, \ldots, a_n\}$. We have a map $A^* \to G$, denoted by $w \mapsto \overline{w}$. G is an **automatic group** if*

1. *there is an FSA M over A such that $\pi : L \to G$ is onto;*
2. *the following languages are regular:*
   - $L_= = \{(u, v) : u, v \in L(M) \text{ and } \overline{u} = \overline{v}\}$
   - $L_{a_1} = \{(u, v) : u, v \in L(M) \text{ and } \overline{u} = \overline{va_1}\}$

$\vdots$

   -
   - $L_{a_n} = \{(u, v) : u, v \in L(M) \text{ and } \overline{u} = \overline{va_n}\}$

This definition is equivalent to the following:
G is an automatic group if

1. there is an FSA M such that $\pi : L(M) \to G$ is onto;
2. there is a constant k such that $u, v \in L(M), d_w(\overline{u}, \overline{v}) = 1$ implies that u and v satisfy the k-fellow traveller property.

Easy examples: finite groups, Z, F_2 (drawn on side board)
Other examples: hyperbolic groups, abelian groups, braid group, mapping class group

Closure properties: direct product, free product, free product with amalgamation over a finite subgroup, finite index subgroups and supergroups, quasiconvex subgroups

Non-examples: $\mathrm{SL}_n(\mathbb{Z}), n \geq 3$, non-silly Baumslag-Solitar groups

Question: nilpotent groups?

3. Lemmas

Lemma 3.1 (short representatives)**.** *Let G be automatic with automatic structure (A, L). There there is a constant N such that any $w \in A^*$ is equivalent to some $w' \in L$ where $|w'| \leq N|w| + n_0$ where n_0 is the length of an accepted representative of 1.*

proof by machines. First we claim that there is a constant M such that if $w \in L, g \in G$ with $d_w(\overline{w}, g) \leq 1$, then there is a representative $u \in L$ with $\overline{u} = g$ such that $|u| \leq |w| + M$. To see this, choose M bigger than the number of states in any FSA in definition I. Let $u \in L$ such that $\overline{u} = g$, so (w, u) or (u, w) is accepted by one of the M_x for some $x \in A \cup \epsilon$. If $|u| > |w| + M$, then M_x undergoes more than M transitions after reading w, so we could shorten u by removing a loop from the path of transitions in M_x.

Now we take $N = M$ and induct on $|w|$. If $|w| = 0$, w is the empty word and so it represents the identity and thus $w \sim w' \in L$ with $w' \in L$ and $w' = n_0 = N|w| + n_0$.

If $|w| > 0$, then $w = ux$ for some $x \in A$ and $|u| < |w|$. Apply IH to u so there is a $u' \sim u$ with $u' \in L$ and $|u'| \leq N(|w| - 1) + n_0$. But the claim implies that there is a word $w' \in L$ with $\overline{w'} = \overline{w}$ and $|w'| \leq |u'| + N \leq N|w| + n_0$. $\square$

Definition 2. *Suppose $w \in L$ represents the trivial element. Then $w = \prod_{i=1}^{n} v_i r_i^{\pm 1} v_i^{-1}$ with $r_i \in R, v_i \in F(A)$. Say that a representation of w has* ***combinatorial area*** *area(w) $= n$. Define $\phi(i) = \max\{\text{area}(w) : |w| \leq i, \overline{w} = 1\}$, the* ***isoperimetric function*** *for $< A|R >$. Note: if ϕ' is an isoperimetric function for a different presentation of the same group, then $\phi(i) \leq C_1\phi(C_2 i)$ for some constants C_1, C_2, so if $\phi(i)$ is bounded by, say, a polynomial, then ϕ' will be bounded by some polynomial as well. For example, if ϕ is bounded by a quadratic polynomial, then we say that G satisfies a* ***quadratic isoperimetric inequality****.*

Lemma 3.2. *Automatic groups have quadratic isoperimetric inequality.*

proof by picture. Goal: given any representative w of the trivial word, we must show that the combinatorial area of some disk D with boundary w has area bounded above by a quadratic polynomial in $|w|$. Take N, n_0 as in Lemma 3.1, and find u_t such that $\overline{u_t} = \overline{w(t)}$ and $|u_t| \leq N|w| + n_0$ for each t. Also for all t, $\overline{u_t}$ and $\overline{u_{t+1}}$ differ by a generator, say x_t. By definition II, $d_w(u_t(s), u_{t+1}(s)) \leq k$ for all t, s, so the loop $u_t x_t u_{t+1}^{-1}$ decomposes into no more than $N|w| + n_0$ loops of length at most $2k + 2$. Therefore D may be decomposed into no more than

$$|w|(N|w| + n_0) = N|w|^2 + n_0|w|$$

disks of length at most $2k + 2$. Consider the presentation $G =< A|R >$ where $R = \{w|\overline{w} = 1, |w| \leq 2k + 2\}$. Then the isoperimetric function with respect to this presentation is bounded by a quadratic polynomial in $|w|$, namely $N|w|^2 + n + 0$. □

Note: compare with hyperbolic groups, which satisfy a linear isoperimetric inequality.

4. The Heisenberg Group

We return to our question of whether nilpotent groups are automatic. It turns out that the answer is negative.

Define $\mathcal{H} =< \alpha, \beta, \gamma|[\alpha, \beta]\gamma^{-1}, [\alpha, \gamma], [\beta, \gamma] >$.

Theorem 4.1. *The Heisenberg group $\mathcal{H}$ is not automatic.*

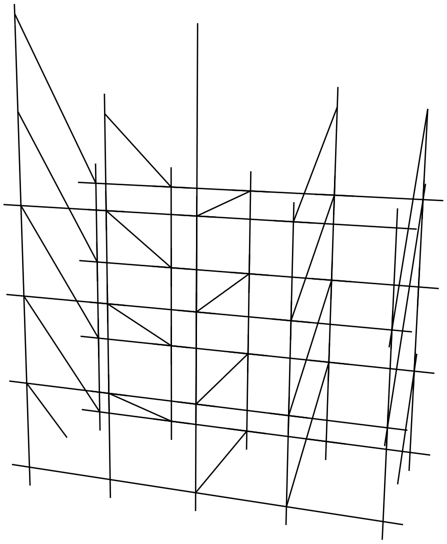
Proof. We begin by observing that

$$\mathcal{H} \cong H = \mathbb{R}^3/G$$

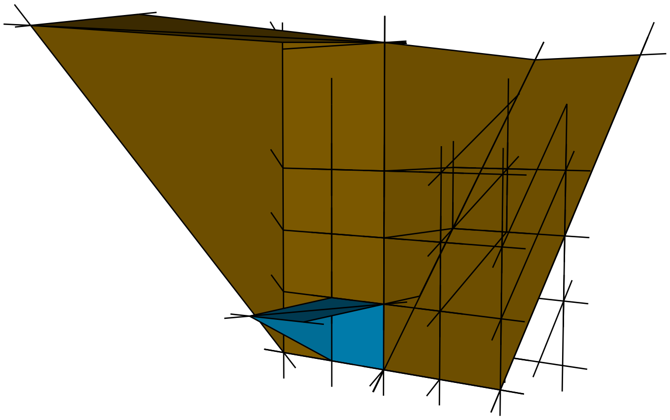
where $G =< \alpha(x, y, z) = (x + 1, y, z + y), \beta(x, y, z) = (x, y + 1, z), \gamma(x, y, z) = (x, y, z + 1) >$ is a group of isometries of $\mathbb{R}^3$. (picture) To see this, note that H is just the HNN extension given by the maps $\iota, \varphi : \mathbb{Z}^2 \to \mathbb{Z}^2$ where ι is inclusion and $\varphi(x) = \alpha, \varphi(y) = \alpha\beta$, so

$$\begin{aligned} H &= < \alpha, \beta, t|[\alpha, \beta] = 1, t\alpha t^{-1} = \alpha, t\beta t^{-1} = \alpha\beta > \\ &= < \alpha, \beta, t|[\alpha, \beta] = 1, [\alpha, t] = 1, [t, \beta] = \alpha > \end{aligned}$$

We now use the action of G on $\mathbb{Z}^3 \in \mathbb{R}^3$ to build the Cayley graph in $\mathbb{R}^3$.



We now consider the word $w_n = \alpha^n \beta^n \alpha^{-n} \beta^{-2n} \alpha^{-n} \beta^n \alpha^n$. We claim that any disk spanning it in Γ must consist of at least n^3 "elementary spanning surfaces", i.e. disks corresponding to conjugates of relators, which are just images of S_1, S_2, S_3 under G, where S_1 spans $[\alpha, \beta]\gamma^{-1}$, S_2 spans $[\alpha, \gamma]$, and S_3 spans $[\beta, \gamma]$.



In order to bound the combinatorial area from below, just estimate the area of spanning disks from below. Projecting to the yz-plane, we find that S_1 has projectional area 1/2, S_2 has projectional area 0, and S_3 has projectional area 1. But the projectional area of D is n^3, so D must consist of at least n^3 elementary spanning disks, and therefore the minimal isoperimetric inequality of H is at least cubic. In particular, H does not satisfy a quadratic isoperimetric inequality, so it cannot be automatic. $\square$

REFERENCES

[Far92] Benson Farb, *Automatic groups: A guided tour*, L'Enseignment Math. **38** (1992), 291–313.