KR HOMOLOGY — RASMUSSEN: "SOME DIFFERENTIALS OF K-R HOMOLOGY."

ERIC ①

JUST ABOUT KNOTS

· **HOMFLY POLYNOMIAL**

$P_K(a,q)$ DEFINED BY

$$aP_K(\text{positive crossing}) - a^{-1}P_K(\text{negative crossing}) = (q-q^{-1})\,P_K(\text{smoothing})$$

$$P(O) = 1.$$

BUILD $H_\cdot(K) \ni q\text{-}\chi(H_\cdot(K)) = \dfrac{P_K(a,q)}{q-q^{-1}}$.

· **sl_n HOMOLOGY** $H_N(K) \ni q\text{-}\chi(H_N(K)) = P_K(q^N, q)$.

CONSTRUCTION MATRIX FACTORISATION:

R comm. RING, $W \in R$.

DEFINE $\mathbb{Z}$ GRADED MATRIX FACTORISATION OVER R WITH POTENTIAL W TO BE THE FREE GRADED R-MODULE C^k ($k \in \mathbb{Z}$) WITH TWO DIFFERENTIALS DENOTED $d_\pm : C^* \to C^{*\pm 2}$ $\ni (d_+ + d_-)^2 = w \cdot I\lambda$.

RKS YOU GET TWO COMPLEXES, $(C^*_\pm, d_\pm)$, AND IF $W=0$, YOU GET A $\mathbb{Z}/2\mathbb{Z}$ GRADED CPLX $(C^*_{tot}, d_+ + d_-)$.

$$\cdots\ C\{j\} \underset{d_-}{\overset{d_+}{\rightleftarrows}} C\{j+2\} \underset{d_-}{\overset{d_+}{\rightleftarrows}} C\{j+4\}\ \cdots$$

GMF(R):

OBJ – GR. MAT. FACT. OVER R

MORPH – HOMS OF GR R MODULES $f: C^{*} \to D^{*}$ THAT COMM. W/ DIFFS.

$GMF_w(R) \subset GMF(R)$

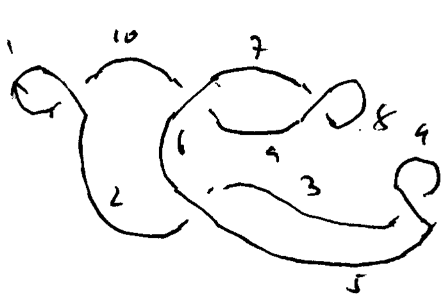
SUBCAT. OF GMF(R) WITH POTENTIAL w.

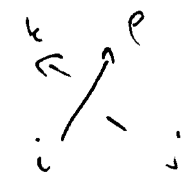
NOW MAKE A COMPLEX OF MF

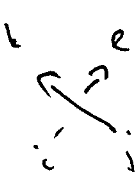
$$\begin{array}{ccccccc} & \uparrow d_v & & \uparrow d_v & & \uparrow d_v & \\ \rightleftarrows & C\{i,j,k\} & \underset{d_-}{\overset{d_+}{\rightleftarrows}} & C\{i,j+2,k\} & \underset{d_-}{\overset{d_+}{\rightleftarrows}} & C\{i,j+4,k\} & \rightleftarrows \\ & \uparrow d_v & & \uparrow d_v & & \uparrow d_v & \\ \rightleftarrows & C\{i,j,k-2\} & \underset{d_-}{\overset{d_+}{\rightleftarrows}} & C\{i,j+2,k-2\} & \underset{d_-}{\overset{d_+}{\rightleftarrows}} & C\{i,j+4,k-2\} & \rightleftarrows \\ & \uparrow d_v & & \uparrow d_v & & \uparrow d_v & \end{array}$$

NOW ADD THIRD GRADING i – FIRST SLOT.

· TANGLE DIAGRAMMES



$D_+ =$ 

$D_- =$ 

D_m: 

Make edge ring $R(D)$:

$$\mathbb{Q}[E_1, \ldots, E_n] \Big/ (p(V_i))$$

Polynomial ring over the edges

$$P(v_+) = X_k + X_l - X_i - X_j$$

(+ for ingoing, − for outgoing)

$$P(v_-) = P(v_+)$$

$$P(v_m) = X_j - X_i \, .$$

$R(D)$ is graded: $q(X_i) = 2 \ \forall i$.

KR Complex

Fix an aux. poly $p(x) \in \mathbb{Q}[x]$,

$C_p(D)$ is a graded module over $R(D)$

→ q grading of $C_p(D)$.

$C_p(D)$ is a complex of matrix factorizations

→ horizontal & vertical gradings

"Koszul" "homological"

gr_h gr_v

$C_p(D_+)$ cx of MF^s over $R = R(D_+) \cong \mathbb{Q}[x_i, x_j, x_k, x_l] \big/ (x_k + x_l - x_i - x_j)$

potential $w_p = p(x_k) + p(x_l) - p(x_i) - p(x_j)$

$$C_p(D_+) \quad \begin{array}{ccc} R\{0,-2,0\} & \overset{x_k - x_i}{\underset{\longleftarrow}{\longrightarrow}} & R\{0,0,0\} \\ x_j - x_k \uparrow & & \uparrow -(x_k - x_i)(x_k - x_j) \quad 1 \\ R\{2,-2,-2\} & \underset{\longleftarrow}{\longrightarrow} & R\{0,0,-2\} \end{array}$$

$C_p(D)$: AT EACH CROSSING FORM

$$C_p(D_c) \otimes_{R(D_c)} R(D)$$

$$C_p(D) := \bigotimes_c \left(C_p(D_c) \otimes_{R(D_c)} R(D) \right)$$

OR ($R(D)$ IF NO CROSSING)

HOMFLY HOMOLOGY THEORY:

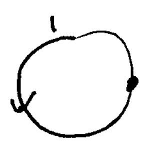
$$H(K) = H\left(H(C_p(D), d_+), d_v^*\right)\{w+b, w+b-1, w-b+1\}$$

w = writhe of D

b = # of strands in (braid) diagram of D.

Thm 1. (KR) $H(K)$ IS A KNOT (LINK) INVARIANT.

2. (KR) $\sum_{ijk} (-1)^{(k-j)/2} a^j q^i \dim H^{i,j,k}(K) = -\frac{P_K(a,q)}{q-q^{-1}}$.

Eg K = UNKNOT D= 

$$R(D) = \mathbb{Q}[x_1]/(x_1 - x_1) \cong \mathbb{Q}[x_1]$$

$$C_p(D) = R(D) = \mathbb{Q}[x_1]$$

$$H(K) \cong \mathbb{Q}[x_1]\{1,0,0\}$$

EX K = UNKNOT  D $\quad w=1$, $b=2$

$$R = R(D) = \mathbb{Q}[x_1, x_2] \big/ (x_2 - x_2 + x_1 - x_1)$$

$$= \mathbb{Q}[x_1, x_2]$$

$$C_p(D_c) = R\{0,-2,0\} \xrightarrow{0} R\{0,0,0\}$$

$$(x_2 - x_1) \uparrow \qquad\qquad \uparrow 1$$

$$R\{2,-2,-2\} \xrightarrow{0} R\{0,0,-2\}$$

$$H(C_p(D_c), d_+) = R\{0,-2,0\} \qquad R\{0,0,0\}$$

$$(x_2 - x_1)\uparrow \qquad\qquad \uparrow 1$$

$$R\{2,-2,-2\} \qquad R\{0,0,-2\}$$

$$H(H(C_p(D_c), d_+), d_v^*) = \mathbb{Q}[x_1, x_2] \big/ {x_2 - x_1} \; \{0,-2,0\}$$

$$= \mathbb{Q}[x_1]\{0,-2,0\}.$$

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sl_n HOMOLOGY $\quad w_p = 0$

$$H_p(K) \doteq H(H(C_p(D)/(x_1), d_{tot}), d_v^*).$$

$$H_N(K) \doteq H_{x^{N+1}}(K)\ \{(N-1)w, 0\}$$

w/ grading (gr_h, gr_v) $\qquad gr_w = q + (N-1)\, gr_h.$

THM $$\sum_{I,J} (-1)^J q^I \dim H_N^{I,J}(K) = P_K(q^N, q)$$

WHERE DOES

$$\begin{array}{ccc} & x_e - x_j & \\ & \parallel & \\ C_p(D_c) = R\{0,-2,0\} & \xrightarrow{x_k - x_i} & R\{0,0,0\} \\ {\scriptstyle x_j - x_k}\uparrow & & \\ R\{2,-2,-2\} & \xrightarrow{-(x_k - x_i)(x_k - x_j)} & R\{0,0,-2\} \end{array}$$

$\longleftarrow$ $H^1(\text{top row}) = 0$

$0 \qquad R/_{x_i = x_k,\ x_l = x_i}$

COME FROM?