Appendix Appendix A. Handling Constants with Care

Our goal for this appendix is to expand some of the arguments involving implicit constants in order to allay any uneasiness that may have been effected by the apparent lack of rigor.

(coming soon...)

Appendix Appendix B. Gowers' Technical Lemma

The objective of this appendix is to provide proofs of Gowers' technical lemma used to prove Proposition 5.2 as well as a proof of Hölder's inequality, which is the only non-elementary result that the proof of that lemma requires.

Much to my surprise, Anindya reproved Gowers' quantatative version of the Balog-Szemeredi theorem in his notes on the Gowers' norm, and in so doing he typed up the technical combinatorial lemma required in Gowers' proof. As promised in Week 5, I am providing it here in the appendix, although you may also find it in Anindya's notes for Week 7.

Lemma B.1. *Let X be a set of size m, and let $A_1, \cdots, A_n$ are subsets of X such that $\sum_{i,j \in [n]} |A_i \cap A_j| \geq \delta^2 mn^2$. Then there is a set $K \subseteq [n]$ of size at least $\delta^5 n/\sqrt{2}$, such that, for at least $16/17$ fraction (or 90%) of the pairs of $(i, j) \in K^2$ $|A_i \cap A_j| \geq \delta^2 m/2$.*

In particular, the result holds if $|A_i| \geq \delta m$ for all $i \in [n]$.

Proof. Let $B_i = \{j | i \in A_j\}$. Define $E_i = B_i^2$. For any given $x, y \in [n]$, We first calculate

$$p_{xy} \stackrel{\text{def}}{=} \Pr_{i \in [m]} [\, (x, y) \in E_i \,] = \frac{|A_x \cap A_y|}{m}.$$

Now choose independently and uniformly randomly $j_1, \cdots, j_5$ and set

$$X = \cap_{k \in [5]} E_{j_k}.$$

Clearly, $\Pr[(x, y) \in X] = p_{xy}^5$. Thus

$$\mathrm{E}X = \sum_{xy} p_{xy}^5.$$

However, since $\sum_{xy} p_{xy} \geq \delta^2 n^2$, and since $\left(\frac{\sum p_{xy}}{n^2}\right)^5 \leq \left(\frac{\sum p_{xy}^5}{n^2}\right)$, we obtain $\mathrm{E}X \geq \delta^{10} n^2$. For the random choice of X, consider the subset $Y \stackrel{\text{def}}{=} \{(i, j) \in X \ : \ |A_i \cap A_j| \leq \delta^2 m/2\}$ (i.e., all (i, j) such that $p_{ij} \leq \delta^2/2$). Clearly then $\mathrm{E}Y \leq (\delta^2/2)^5 n^2$. Thus $\mathrm{E}|X - 16Y| \geq \delta^{10} n^2/2$.

Thus there exists a choice of j_r such that $|X| \geq 16|Y|$ and $\mathrm{E}X \geq \delta^1 0 n^2/2$. Set $K^2 = X$ (i.e., $K \stackrel{\text{def}}{=} \cap B_{j_r}$).

For the second statement, let $s_i = |B_i|$. Then note $\frac{\sum_i s_i^2}{m} \geq \left(\frac{\sum_i s_i}{m}\right)^2 \geq \left(\frac{\delta m \cdot n}{m}\right)^2 \geq \delta^2 n^2$.

□

Finally, to complete the proof of the BKT theorem, we present the following theorem, well-known from analysis:

Theorem B.2 (Hölder's Inequality)**.**