# L' Hospital's Role

L' Hospital's Role is a method for evaluating indeterminate forms like $\frac{0}{0}$ and $\frac{\infty}{\infty}$

Def: $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(a)}{g'(a)}$

Conditions
1) $f(x)$ and $g(x)$ must exist and be differentiable
2) $g'(x) \neq 0$

Example: $\lim\limits_{x \to 0} \frac{\sin x}{x} = \frac{0}{0}$ indet

$$\overset{*}{=} \lim_{x \to 0} \frac{\cos x}{1} = 1$$

NOTE: For indeterminate forms like $\infty - \infty$ or $0 \cdot \infty$, transform the limit into $\frac{f(x)}{g(x)}$ so L' Hospital's Role can be applied.

L' Hospital's Role is only for indeterminate forms.

Remember! L' Hospital's Role is only for indeterminate forms.

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{2} = 0$$

but had we used L' Hospital's...

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \lim_{x \to \pi^-} \frac{\cos x}{\sin x} = -\infty$$

PRACTICE

1) $\lim\limits_{x \to 1} \frac{\ln x}{\sin \pi x} = \frac{0}{0}$ indet.

$$\overset{*}{=} \lim_{x \to 1} \frac{\frac{1}{x}}{\pi \cos \pi x} = -\frac{1}{\pi}$$

2) $\lim\limits_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{0}{0}$ indet.

$$\overset{*}{=} \lim_{x \to 0} \frac{\sin x}{2x} \overset{*}{=} \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$$

3) $\lim\limits_{x \to \infty} \frac{(\ln x)^2}{x} = \frac{\infty}{\infty}$ indet

$$\overset{*}{=} \lim_{x \to \infty} \frac{2(\ln x)\frac{1}{x}}{1} = \lim_{x \to \infty} \frac{\frac{2 \ln x}{x}}{1} = \infty$$

$$\overset{*}{=} \lim_{x \to \infty} \frac{2}{x} = 0$$

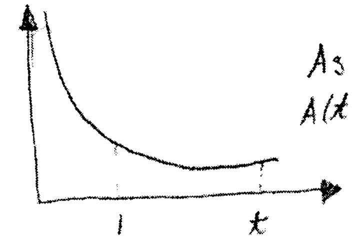
# Improper Integrals

Def: A definite integral w/ either an infinite interval or an infinitely discontinuous function.

## Type 1 - Infinite Intervals

a) If $\int_a^t f(x)\,dx$ exists for all $t \ge a$ then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$

b) If $\int_t^b f(x)\,dx$ exists for all $t \le b$ then $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$




As $x \to \infty$
$A(t) \to 1$

c) If $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ exists they are convergent. Likewise if they don't exist, they are divergent.

i) If both $\int_c^\infty f(x)\,dx$ and $\int_{-\infty}^c f(x)\,dx$ are convergent then $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$

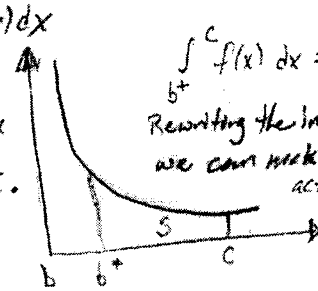
Note: For a), b) and c) the limits must be finite numbers.

## Type 2 - Infinite Discontinuity

a) If $f$ is continuous on $[a,b)$ and disc. at b, $\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$

b) If $f$ is continuous on $(a,b]$ and disc. at a, $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$

c) $\int_a^b f(x)\,dx$ is conv. if the limit exists and divergent if it DNE.




$\int_{b^+}^c f(x)\,dx = S$

Rewriting the integral as a limit, we can make S approach the actual area from a discontinuous point.

i) If $f$ is disc. at c where $a < c < b$ and $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

## P Function

$\int_1^\infty \frac{1}{x^p}\,dx$ is conv. for $p > 1$ and divergent for $p \le 1$

## Comparison Test for Improper Integrals

$f$ and $g$ are continuous functions where $f(x) \ge g(x) \ge 0$ for $x \ge a$

$\int_a^\infty f(x)\,dx$ is conv. then $\int_a^\infty g(x)$ is conv. ; $\int_a^\infty g(x)\,dx$ is div then $\int_a^\infty f(x)\,dx$ is divergent

## Practice

1) $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$ $\quad u = x^2+2 \quad \frac{du}{2} = x\,dx$

$\frac{1}{2}\int_0^\infty u^{-2}\,dx = \lim_{t\to\infty} \frac{1}{2}\int_0^t u^{-2}\,dx = \lim_{t\to\infty} \frac{1}{2}\left[u^{-1}\right]_0^t$

$= \lim_{t\to\infty} \frac{1}{2}\left[\frac{1}{(x^2+2)}\right]_0^t = -\frac{1}{2}\left[0 - \frac{1}{2}\right] = \boxed{1/4}$

2) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$\lim_{t\to1^-} \int_0^t \frac{dx}{\sqrt{1-x^2}}$

$= \lim_{t\to1^-} \sin^{-1} x \Big]_0^t = \boxed{\pi/2}$

**Def.** A sequence $\{a_n\}$ has the limit $L$ & we write

$$\lim_{n\to\infty} = L \quad \text{or} \quad a_n \to L \text{ as } n\to\infty$$

if we can make the terms $a_n$ as close to $L$ as we like by taking $n$ sufficiently large. If $\lim_{n\to\infty} a_n$ exists we say the seq. converges otherwise it is divergent

**Def.** A sequence $\{a_n\}$ has the limit $L$ & we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n\to\infty$$

if for every $\varepsilon > 0$ there is a corresponding integer $N$ such that

$$|a_n - L| < \varepsilon \quad \text{whenever} \quad n > N$$

**Theorem** If $\lim_{x\to\infty} f(x) = L$ & $f(n) = a_n$, when ~~x~~ $n$ is an integer, then

$$\lim_{n\to\infty} a_n = L$$

**Def.** $\lim_{n\to\infty} a_n = \infty$ means that for every positive number $M$ there is an integer $N$ such that $a_n > M$ whenever $n > N$

If $\{a_n\}$ & $\{b_n\}$ are convergent sequences & $c$ is constant then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n * \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if} \quad \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \text{ if } p > 0 \text{ \& } a_n > 0$$

**Theorem** If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ & divergent for all other values of $r$

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

**Def** A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$. It is decreasing if $a_n > a_{n+1}$ for all $n \geq 1$

It is called monotonic if it is either increasing or decreasing.

**Def.** A sequence $\{a_n\}$ is bounded above if there is a number $M$ such that $a_n \leq M$ for all $n \geq 1$

It is bounded below if there is a number $m$ such that

$m \leq a_n$ for all $n \geq 1$

If it is bounded above & below, then $\{a_n\}$ is a bounded sequence

**Monotonic Seq. Theorem**

Every bounded, monotonic seq. is convergent

**Squeeze theorem** If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ & $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$

**Ex 1.**

Find $\lim_{n\to\infty} \frac{n}{n+1}$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}}$$

$$= \frac{1}{1+0} = 1$$

**Def.** Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$, let $s_n$ denote its $n$th partial sum

$$s_n = \sum_{n=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

If the seq. $\{s_n\}$ is convergent & $\lim_{n\to\infty} s_n = S$ exists as a real number, then the series $\sum a_n$ is called convergent & we write

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = S \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = S$$

The number $s$ is called the sum of the series. Otherwise the series is divergent.

**Def.** The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots$$ is convergent if $|r| < 1$ & its

sum is $\sum_{n=1}^{\infty} ar^{n-1} = \dfrac{a}{1-r}$ $|r| < 1$

if $|r| \geq 1$, the geometric series is divergent

**Theorem** If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

**Test for Divergence** If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent

**Theorem** If $\sum a_n$ & $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where $c$ is constant), $\sum(a_n + b_n)$ & $\sum(a_n - b_n)$, &

(i) $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

# Section 12.3

## The Integral Test

**1) Theorems**

a) The Integral Test

i) If $f$ is continuous, positive, and decreasing and $f(x) = A_n$ then

(1) If $\int^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} A_n$ is continuous

(2) If $\int^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} A_n$ is discontinuous

b) P-series

i) $\sum_{n=1}^{\infty} 1/n^p$ is convergent if p >1 and divergent if p $\leq$ 1

**2) Proofs**

a) Proof of p-series using Integral Test

i) $\sum_{n=1}^{\infty} 1/n^p = \int^{\infty} 1/x^p \, dx$

$$\int^{\infty} 1/x^p \, dx = \lim_{b \to \infty} \int^{b} x^{-p} dx$$

$$= \lim_{b \to \infty} \frac{x^{1-p}}{1-p}\bigg|_{x=1}^{x=b}$$

$$= \lim_{b \to \infty} \frac{1}{1-p}\left[\frac{1}{b^{p-1}} - 1\right]$$

So if p<1, the series converges and if p $\geq$ 1, the series diverges

# Section 12.3

## The Integral Test

Example 1: Determine rather the series $\sum_{n=1}^{\infty}\frac{\ln n}{n}$ converges or diverges

$$\sum_{n=1}^{\infty}\frac{\ln n}{n} = \int^{\infty}\frac{\ln x}{x}dx$$

$$=\lim_{b\to\infty}\int^{b}\frac{\ln x}{x}dx \qquad u=\ln x$$

$$=\lim_{b\to\infty}\int^{b}udu \qquad du=\frac{dx}{x}$$

$$=\lim_{b\to\infty}\frac{u^2}{2}\Bigg]_1^b$$

$$=\lim_{b\to\infty}\frac{\ln x^2}{2}\Bigg]_1^b$$

$$=\frac{\ln\infty^2}{2}-\frac{\ln 1^2}{2}$$

$$=\infty$$

Integral diverges, so series diverges due to the integral test.

Example 2: Using integral test, prove that $\sum_{n=1}^{\infty}\frac{1}{n^4}$ converges

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \int^{\infty}\frac{1}{x^4}dx$$

$$=\lim_{b\to\infty}\int^{b}x^{-4}dx$$

$$=\lim_{b\to\infty}\frac{-x^{-3}}{3}\Bigg|_1^b$$

$$=\lim_{b\to\infty}\frac{-1}{3b^3}+\frac{1}{3}$$

$$=\frac{1}{3}$$

The integral converges, so the series converges due to the integral test, thus proving p-series

# Section 12.4
## Comparison Tests

**1) Theorems**

a) The Comparison Test

i) If $\sum A_n$ and $\sum B_n$ both have positive terms and $\sum B_n \geq \sum A_n$ for all n. then,

(1) If $\sum B_n$ is convergent, then $\sum A_n$ is also convergent

(2) If $\sum A_n$ is divergent, then $\sum B_n$ is also divergent

b) The Limit Comparison Tests

i) If $\sum A_n$ and $\sum B_n$ both have positive terms and $\lim_{n\to\infty} \frac{A_n}{B_n} = c$ where $c$ is finite and positive, then both series converge or they both diverge.

**2) Proofs**

a) The Comparison Test

i) The comparison test can be thought of as a squeeze theorem with the lowest term at 0.

b) The Limit Comparison Test

i) Assuming $c$ is greater than $m$ and less than $M$ and since $\lim_{n\to\infty} \frac{A_n}{B_n} = c$

we can assume….

$m < \frac{A_n}{B_n} < M = mB_n < A_n < MB_n$

In this case, if $\sum B_n$ converges, so does $\sum MB_n$ and $\sum A_n$ would converge due to the comparison test. If $\sum B_n$ diverges, so does $\sum mB_n$ and $\sum A_n$ diverges due to comparison test.

# 12.5 Alternating Series

An <u>alternating series</u> is a series whose terms are alternately positive and negative.

ex// $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$a_n = (-1)^{n-1} \cdot b_n \qquad \text{where} \qquad b_n = |a_n|$$

## Alternating Series Test

examine $$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot b_n$$

- this series will converge if it meets the following 2 conditions:

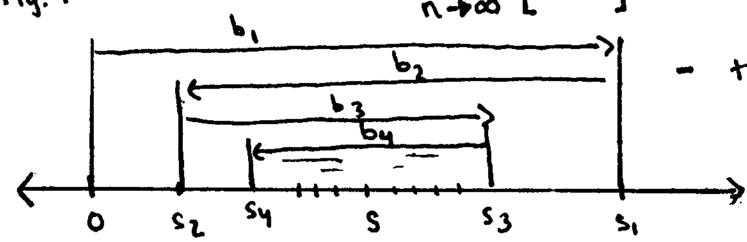
1) $b_{n+1} \leq b_n$ for all $n$

   - that is, if each preceeding term is greater than ... in magnitude.

2) $\lim\limits_{n \to \infty} \left[ b_n \right] = 0$

   - the terms in the series head to 0.

Fig. 1




They ultimately approach a sum (S)

In Fig. 1, you can see that the partial sums are oscillating back and forth increasing + decreasing w/ smaller intervals.

## Estimating Sums

The partial sum ($s_n$) of any convergent series can be used to approximate the total sum ($s$). The accuracy of the approximation can be calculated as follows:

$$\text{let} \quad s = \sum (-1)^{n-1} \cdot b_n$$

(sum of an alternating series)

* if $0 \leq b_{n+1} \leq b_n$

and $\lim_{n \to \infty} [b_n] = 0$

then the remainder ($R_n$) is equal to $(s - s_n)$

$$|R_n| = |s - s_n| \leq b_{n+1}$$

The remainder ($R_n$) will be the size of error, how far off the estimate is, from the actual sum; and it will be no greater than the absolute value of the first neglected term, ($b_{n+1}$).

* for Alternating Series only

Using this estimation technique we can decide/calculate the number of terms needed to get a certain accuracy.

(1)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$$ Test for convergence or divergence.

The series is alternating, so test for convergence:

$$a_n = \frac{(-1)^{n-1}}{\sqrt{n}} \qquad a_{n+1} = \left| \frac{(-1)}{\sqrt{n+1}} \right|$$

$$|a_{n+1}| \overset{?}{\leq} |a_n|$$

$$\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} \quad \checkmark$$

$$\lim_{n \to \infty} \left[ \frac{1}{n^{1/2}} \right] = 0 \quad \checkmark$$

$\therefore$ This series $\boxed{\text{converges}}$ b/c it satisfies both conditions of the AST, that

$|a_{n+1}| \leq |a_n|$ and that $\lim_{n \to \infty} \left[ |a_n| \right] = 0$.

(6)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2+1}$$ Test for convergence or divergence.

Series alternates ✓

$$|a_n| = \frac{1}{4n^2+1} \qquad |a_{n+1}| = \frac{1}{4(n+1)^2+1}$$

$$= \frac{1}{4(n^2+2n+1)+1}$$

$$= \frac{1}{4n^2+8n+5}$$

$$|a_{n+1}| \overset{?}{\leq} |a_n|$$

$$\frac{1}{4n^2+8n+5} \leq \frac{1}{4n^2+1}$$ for all n ✓

$$\lim_{n\to\infty} \left[ \frac{1}{4n^2+1} \right] = 0$$ ✓

∴ This series $\boxed{\text{converges}}$ by the Alternating Series Test

(5)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$$

Approximate sum of series correct to 4 decimal places

Firstly, does this series converge?

<u>Alternating Series Test</u>

$$|a_{n+1}| \overset{?}{\leq} |a_n|$$

$$\frac{1}{(n+1)^5} \leq \frac{1}{n^5} \quad \checkmark$$

$$\lim_{n\to\infty} \left[ \frac{1}{n^5} \right] = 0 \quad \checkmark$$

$\therefore$ This series converges and we can approximate the sum to 4 decimal places.

Since it met the 2 above conditions, we can apply the Alternating Series Estimation Theorem

$R_n$ (the remainder, size of error) is equal to the absolute value of the first ignored term; we thus, need to find the 1st term w/ an absolute value of $\frac{1}{10^4}$ (at most).

$$\overset{a_1}{1} - \frac{1}{32} + \overset{a_3}{\frac{1}{243}} - \frac{1}{1024} + \overset{a_5}{\frac{1}{3125}} - \frac{1}{7776} + \overset{a_7}{\frac{1}{16807}} + \dots$$

# 12.6 - ABSOLUTE CONVERGENCE AND ROOT AND RATIO TESTS

- **ABSOLUTE CONVERGENCE**

A SERIES WHERE THE ABSOLUTE VALUES OF $\sum |a_n|$ IS CONVERGENT.

EXAMPLE

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$$

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ = CONVERGENT BY ALTERNATING SERIES

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^3} \right| = \frac{1}{n^3}$ = CONVERGENT P-SERIES WITH $p = 3$

$\therefore$ ABSOLUTELY CONVERGENT

- **CONDITIONALLY CONVERGENT**

A SERIES THAT IS CONVERGENT, BUT NOT ABSOLUTELY CONVERGENT. IF A SERIES IS ABSOLUTELY CONVERGENT, IT IS CONVERGENT.

EXAMPLE

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ = CONVERGENT BY ALTERNATING SERIES

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \frac{1}{n}$ = DIVERGENT; HARMONIC SERIES

$\therefore$ CONDITIONALLY CONVERGENT

- **RATIO TEST**

THE SERIES $\sum_{n=1}^{\infty} a_n$ CONVERGES IF:

i) $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

THE SERIES $\sum_{n=1}^{\infty} a_n$ DIVERGES IF:

i) $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$

THE RATIO TEST IS INCONCLUSIVE IF:

i) $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

EXAMPLE

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

$$\lim_{n \to \infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{(n!)}{2^n}$$

$$= \frac{\cancel{2^n}\cancel{(n!)}(2)}{(n+1)\cancel{(n!)}\cancel{2^n}}$$

$$= \lim_{n \to \infty} \frac{2}{n+1}$$

$$= 0 < 1$$

$\therefore$ CONVERGES

EXAMPLE

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$$

$$(a_n)^{1/n} = \left(\frac{e^{2n}}{n^n}\right)^{1/n}$$

$$= \frac{e^2}{n}$$

$$\lim_{n \to \infty} \frac{e^2}{n}$$

$$= 0 < 1$$

$\therefore$ CONVERGES

- **THE ROOT TEST**

GIVEN THE SERIES $\sum_{n=1}^{\infty} a_n$:

1) $\sum_{n=1}^{\infty} a_n$ CONVERGES IF $\lim_{n \to \infty} |a_n|^{1/n} < 1$

2) $\sum_{n=1}^{\infty} a_n$ DIVERGES IF $\lim_{n \to \infty} |a_n|^{1/n} > 1$

3) $\sum_{n=1}^{\infty}$ CAN CONVERGE OR DIVERGE IF $\lim_{n \to \infty} |a_n|^{1/n} = 1$