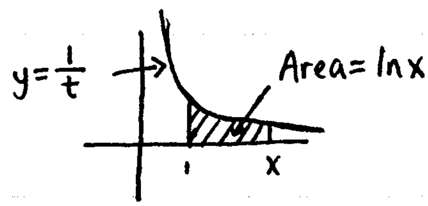
7.2* The Natural Logrithmic Function

Definitions:

1. The natural logrithmic function is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

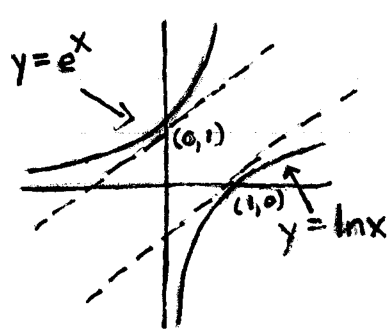
where $x > 0$



2. The inverse function of the natural logarithmic function, $\ln x = y$, is $e^x = y$.
3. The natural logarithmic function of some number, a, can also be defined as

$$\ln a = \lim_{h\to 0} \frac{a^h - 1}{h}$$

4. e is the number such that $\ln e = \lim_{h\to 0} \frac{e^h - 1}{h} = 1$, where $e = a$ in the function above. Geometrically, this means that of all possible exponential functions, $y = a^x$, and logarithmic functions, $y = \log_a x$, the functions $y = e^x$ and $y = \ln x$, respectively, are the ones whose tangent lines at $(0,1)$ and $(1,0)$, ~~respectiv~~ respectively, have a slope of 1.



7.2* The Natural Logarithmic Function (cont.)

Theorems:

1. If $\ln x = y$, then $x = e^y$
2. $\ln x + \ln y = \ln(x \cdot y)$
3. $\ln x - \ln y = \ln\left(\frac{x}{y}\right)$
4. $\ln(x^a) = a \cdot \ln x$
5. $\ln(e^x) = x$
6. $e^{\ln x} = x$ when $x > 0$
7. $\lim_{x \to \infty} \ln x = \infty$ & $\lim_{x \to 0^+} \ln x = -\infty$

Examples:

1. Simplify $\ln 2 + \ln 32$

$$\ln 2 + \ln 32 = \ln(2 \cdot 32) = \ln 64$$

2. Simplify $\ln 80 - \ln 5$

$$\ln 80 - \ln 5 = \ln\left(\frac{80}{5}\right) = \ln 16$$

3. Evaluate $e^{(5-3x)} = 10$

$$e^{(5-3x)} = 10$$
$$\ln(e^{(5-3x)}) = \ln 10$$
$$5 - 3x = \ln 10$$
$$x = \frac{1}{3} \cdot (5 - \ln 10)$$

4. Evaluate $\ln x = 5$

if $\ln x = 5$, then $x = e^5$

7.4 Derivatives of Logarithmic Functions

Definitions:

1. Logarithmic Differentiation is the calculation of derivatives of complex functions involving products, quotients, or powers by using logarithms to first simplify the functions.
2. Steps in Logarithmic Differentiation (see example 5)
   - Take logarithms of both sides of an equation $y=f(x)$, and use properties of logarithms to simplify.
   - Differentiate implicitly with respect to x.
   - Solve resulting equations in terms of y'.

Theorems:

1. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
2. $\frac{d}{dx}(\ln u) = \frac{du}{u\,dx}$
3. $\int \frac{1}{x}\,dx = \ln|x| + C$
4. $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$
5. $\frac{d}{dx}(a^{f(x)}) = a^{x} \cdot \ln a \cdot f'(x)$

Examples:

1. Differentiate $y=\ln(x^3+1)$

   let $u = x^3+1$ and $du = 3x^2dx$

   $y = \ln u$

   $dy = \frac{1}{u}du = \frac{1}{x^3+1} \cdot 3x^2 = \frac{3x^2}{x^3+1}$

2. Differentiate $y = \ln(\sin x)$

   let $u = \sin x$ and $du = \cos x\,dx$

   $y = \ln u$

   $dy = \frac{1}{u}du = \frac{1}{\sin x} \cdot \cos x = \cot x$

7.4 Derivatives of Logarithmic Functions (cont.)

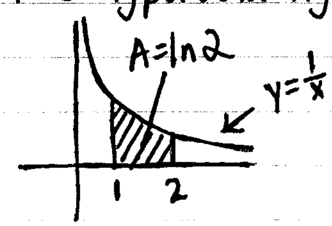
Examples (cont.):

3. Find $\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}}$

$$\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}\left[\ln(x+1) - \frac{1}{2}\ln(x-2)\right]$$
$$= \frac{1}{x+1} - \frac{1}{2x-4}$$
$$= \frac{x-5}{2x^2-2x-4}$$

4. Find the area of the region under the hyperbola $xy=1$ from $x=1$ to $x=2$.

$$A = \int_1^2 \frac{1}{x}\,dx = \ln x\Big]_1^2$$
$$= \ln 2 - \ln 1 = \ln 2 \approx 0.69$$



5. Differentiate $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\left(\frac{1}{x}\right) + \frac{1}{2}\left(\frac{2x}{x^2+1}\right) - 5\left(\frac{3}{3x+2}\right)$$

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

$$\star\ \frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

★ This may seem like a large function, but if we did not use logarithmic differentiation we would have had to use both the Quotient Rule and the Product Rule resulting in a horrendous calculation.

Exponential Functions

General form of an exponential function:

$$f(x)=a^x$$

where a is a positive constant.

$x=n$, $a^n=\underbrace{a\cdot a\cdot a\cdot a\ldots a}_{n \text{ times}}$, where n is a positive integer.

$a^0=1$ when $x=0$ when $x=-n$ $a^{-n}=\frac{1}{a^n}$

If x is a rational number, $x=p/q$, where p & q are integers and $q>0$, then

$$a^x=a^{p/q}=\sqrt[q]{a^p}=(\sqrt[q]{a})^p$$

In general, if a is any positive number, we define

$$a^x=\lim_{r\to x} \lim a^r \quad \text{"}r\text{" rational}$$

– Laws of Exponents:

If $a>0$ and $a\neq 1$, then $f(x)=a^x$ is a continuous function with domain $\mathbb{R}$ and range $(0,\infty)$. In particular, $a^x>0$ for all x.

• $a^{x+y}=a^x a^y$	• $(a^x)^y=a^{xy}$
• $a^{x-y}=\frac{a^x}{a^y}$	• $(ab)^x=a^x b^x$

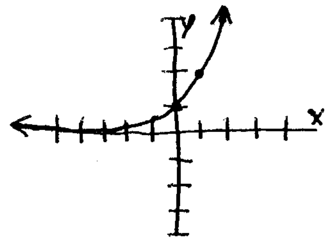
Some limits –

If $a>1$, then $\lim_{x\to\infty} a^x=\infty$ and $\lim_{x\to-\infty} a^x=0$

If $0<a<1$, then $\lim_{x\to\infty} a^x=0$ and $\lim_{x\to-\infty} a^x=\infty$

Exponential Functions (continued)

Ex. Graph of $y=2^x$:



Ex 2. Simplify: $y = x \cdot x^{(5-7)} = x \cdot x^{-2} = \frac{\cancel{x}}{x^{\cancel{2}}} = \frac{1}{x}$

– Definition of the number "e"

$$\boxed{\begin{array}{l} \ln(x) = \int_1^x \frac{1}{t}\,dt = 1 \quad \text{when } x = e \\ \text{as a limit: } e \text{ is the number such that } \lim_{h\to 0} \frac{e^h - 1}{h} = 1 \end{array}}$$

$e \approx 2.718281828459045\ldots$ (it's irrational)

"e" has an interesting property, in that

$\frac{d}{dx} e^x = e^x$ $\quad$ e^x is its own derivative!

Ex. Differentiate $y = e^{\tan x}$

– using the chain rule, we let $u = \tan x$, & have $y = e^u$, so...

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \cdot \frac{du}{dx} = e^{\tan x} \sec^2 x$$

Therefore.. $\boxed{\frac{d}{dx} e^u = e^u \frac{du}{dx}}$

Some properties of the natural exponential function:

$\lim_{x\to -\infty} e^x = 0$ $\qquad$ $\lim_{x\to\infty} e^x = \infty$

So, the x-axis is a horizontal asymptote of $f(x) = e^x$

Exponential Functions
(continued)

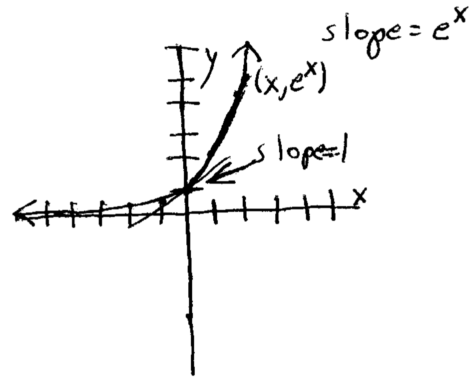
Ex:

Find the derivative of $f(x) = \frac{3e^{3x}}{e^x}$

$f(x) = \frac{3e^{3x}}{e^x} = 3e^{(3x-x)} = 3e^{2x}$

$f'(x) = 6e^{2x}$

A graph of $e^x = f(x)$



The integral of e^x:

$$\boxed{\int e^x dx = e^x + c}$$

Ex. Evaluate $\int x^2 e^{x^3} dx$

- We substitute $u = x^3$. Then $du = 3x^2 dx$, so $x^2 dx = \frac{1}{3} du$...

Solution: $\int x^2 e^{x^3} dx = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + c = \frac{1}{3}e^{x^3} + c$

Inverse Functions

Definitions

- A function f is called a one-to-one function if it never takes on the same value twice.
- Function is one-to-one if and only if no horizontal line intersects its graph more than once.
- For any one-to-one function f with domain A and range B, there is an inverse function f^{-1} with domain B and range A. This is defined by:

$$f^{-1}(y) = x \iff f(x) = y$$

- If $F(x) = (f \circ g)(x) = f[g(x)]$ then $F'(x) = f'[g(x)]\, g'(x)$
- What does it mean in terms of composition to say that f and g are inverse functions of each other?

$$f[g(x)] = x \text{ for all } x \text{ and } g[f(x)] = x$$

- How to find the inverse function of a one-to-one function f:
  1. Write $y = f(x)$
  2. Solve the equation for x in terms of y (if possible).
  3. To express f^{-1} as a function of x, interchange x and y. Resulting equation is $y = f^{-1}(x)$

- If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.
- If f is a one-to-one differentiable function with inverse function $g = f^{-1}$ and $f'[g(a)] \neq 0$, then the inverse function is differentiable at a and

$$g'(a) = \frac{1}{f'[g(a)]}$$

- The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Inverse Trig Functions

- $\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ — $\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$

  $\sin(\sin^{-1} x) = x$ for $-1 \le x \le 1$ — $\cos(\cos^{-1} x) = x$ for $-1 \le x \le 1$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

EXAMPLE PROBLEMS

1. FIND THE INVERSE FUNCTION OF $f(x) = x^3 + 2$

$y = x^3 + 2$

LVE FOR X: $x^3 = y - 2$

$x = \sqrt[3]{y-2}$

INTERCHANGE X AND Y: $y = \sqrt[3]{x-2}$

THEREFORE, $f^{-1}(x) = \sqrt[3]{x-2}$

2. IF f AND g ARE INVERSE FUNCTIONS OF EACH OTHER AND $f'(x)$ IS GIVEN (KNOWN), EXPRESS $g'(x)$ IN TERMS OF f, g, AND f'.

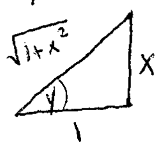
$x = F(x) = f[g(x)]$

$1 = F'(x) = f'[g(x)] \cdot g'(x)$

$$g'(x) = \frac{1}{f'[g(x)]}$$

3. SIMPLIFY THE EXPRESSION $\cos(\tan^{-1} x)$.

IF $y = \tan^{-1} x$, THEN $\tan y = x$, AND WE CAN SEE FROM THE DIAGRAM THAT:



$$\cos(\tan^{-1} x) = \cos y = \frac{1}{\sqrt{1+x^2}}$$

4. FIND THE DERIVATIVE OF $\text{Arcsin}\, x$.

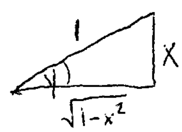
$x = \sin[\text{Arcsin}(x)]$

$1 dx = \cos[\text{Arcsin}(x)] \cdot \text{Arcsin}'(x)$

$$\text{Arcsin}'(x) = \frac{1}{\cos[\text{Arcsin}(x)]}$$

SIMPLIFY $\cos[\text{Arcsin}(x)]$

$y = \sin^{-1} x$; $\sin y = x$



$\cos[\sin^{-1}(x)] = \cos(y) = \sqrt{1-x^2}$

$$\boxed{\text{Arcsin}'(x) = \frac{1}{\sqrt{1-x^2}}}$$

5. $\int \frac{x}{x^4+9}\, dx \quad u = x^2 \quad du = 2x\, dx$

$$\frac{1}{2}\int \frac{du}{u^2+9} = \frac{1}{2} \cdot \frac{1}{3} \text{Arctan}\left(\frac{u}{3}\right) + c$$

$$\boxed{\int \frac{x}{x^4+9}\, dx = \frac{1}{6} \text{Arctan}\left(\frac{x^2}{3}\right) + c}$$