ESP Workshop, Worksheet #10
Thursday October 5, 2006
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1. (a) Compute the following determinants:

$$\text{(a)} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \qquad \text{(b)} \begin{vmatrix} 2 & 1 & -1 \\ 3 & 1 & 0 \\ 4 & -5 & 2 \end{vmatrix} \qquad \text{(c)} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 0 \\ 4 & -5 & 2 \end{vmatrix}$$

   (b) Can you find two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \times \mathbf{b}$ is the answer to 1(c)?

   (c) What is the unit vector pointing in the same direction as $\mathbf{a} \times \mathbf{b}$?

   (d) Find two unit vectors that are orthogonal to both $\langle 1, -1, 1 \rangle$ and $\langle 0, 3, 3 \rangle$.

2. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors and let a and b be real numbers. Determine which of the following are always true, and which are not always true. If possible, give an example where the statement is not true.

   (a) $(a\mathbf{u}) \cdot (b\mathbf{v}) = (ab)\mathbf{u} \cdot \mathbf{v}$

   (b) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

   (c) $\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v}$

   (d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{w} = (\mathbf{w} \cdot \mathbf{u})\mathbf{v}$

   (e) $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|$

   (f) $(\mathbf{u} - \mathbf{v}) - \mathbf{w} = \mathbf{u} - (\mathbf{v} - \mathbf{w})$

   (g) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

   (h) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{v} \times \mathbf{u}\|$

   (i) $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$

   (j) $a\mathbf{u} \times b\mathbf{v} = ab\mathbf{u} \times \mathbf{v}$

3. Prove that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$.

4. (A Chemistry problem!) A molecule of methane, CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The *bond angle* is the angle formed by the H–C–H combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about 109.5 degrees. [*Hint:* Take the vertices of the tetrahedron to be the points $Q = (1, 0, 0)$, $R = (0, 1, 0)$, $P = (0, 0, 1)$, and $S = (1, 1, 1)$ as in Figure 1 (on the back). Then the centroid is (1/2, 1/2, 1/2).]

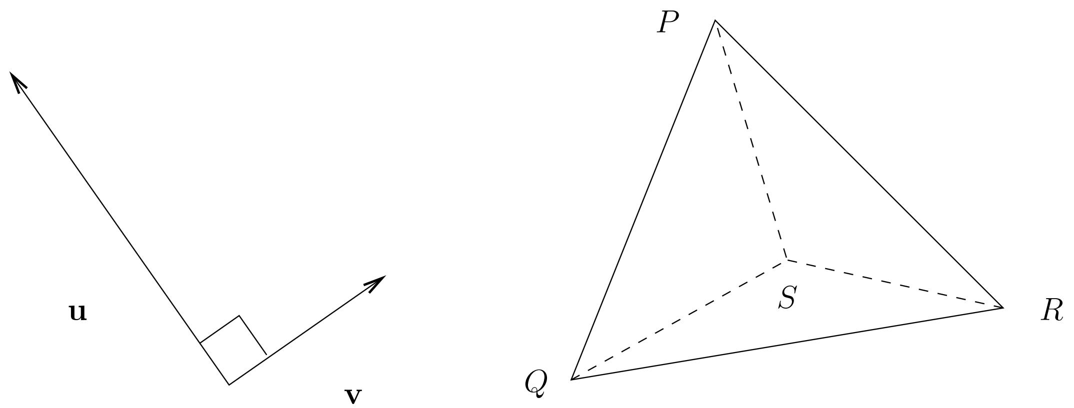

Figure 1: The vectors on the left are for question 5, and the tetrahedron on the right is for questions 4 and 6.

5. Suppose that $\|\mathbf{u}\| = 6$ and $\|\mathbf{v}\| = 3$. Is $\mathbf{u} \times \mathbf{v}$ a vector pointing into the page or out of the page? Also, what is $\|\mathbf{u} \times \mathbf{v}\|$?

6. (The geometry of a tetrahedron.) A tetrahedron is a 3-dimensional solid with four vertices, P, Q, R, and S, and four triangular faces as shown in the figure.

   (a) Let $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$, and $\mathbf{v_4}$ be vectors with lengths equal to the areas of the faces opposite the vertices P, Q, R, and S respectively, and directions perpendicular to the respective faces pointing outward. Show that

   $$\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3} + \mathbf{v_4} = 0$$

   (b) The volume V of a tetrahedron is one-third the distance from a vertex to the opposite face, times the area of that face.

      i. Find a formula for the volume of a tetrahedron in terms of the coordinates of its vertices, P, Q, R, and S.

      ii. Find the volume of the tetrahedron whose vertices are $P = (1, 1, 1), Q = (1, 2, 3), R = (1, 1, 2)$, and $S = (3, -1, 2)$.

   (c) Suppose that the tetrahedron in the figure has a trirectangular vertex S. (This means that the three angles at S are all right angles.) Let A, B, and C be the areas of the three faces that meet at S, and let D be the area of the opposite face PQR. Using the result of Problem 1, or otherwise, show that

   $$D^2 = A^2 + B^2 + C^2$$

   This is a three-dimensional version of the Pythagorean Theorem. Can you think of a three-dimensional version of Fermat's Last Theorem? Can you find any integer solutions to such an equation??