ESP Workshop, Worksheet #15
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1. Match the function with its graph (labeled A–F) and with its contour map (labeled I–IV). Give reasons for your choices.

2. Consider the curve C in $\mathbb{R}^3$ defined by the vector-valued function $\mathbf{r} = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \tan t\mathbf{k}$ on the interval $I = (-\pi/2, \pi/2)$.

   (a) Find the unit tangent vector $\mathbf{T}(t)$ at the point where $t = \pi/4$.

   (b) Write $\mathbf{r}(t)$ parametrically, i.e. in terms of x, y, and z. Find parametric equations for the tangent line at the point from part (a).

   (c) A curve given by a vector function $\mathbf{r}(t)$ on an interval I is called **smooth** if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq 0$ (except possibly on the endpoints of I). Is C smooth on I?

3. Now instead of vector-valued functions $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, we will work with a real-valued function of several variables, $f : \mathbb{R}^2 \to \mathbb{R}$. Let $f(x, y) = \sqrt{x + y}$.

   (a) What is the domain of f? Sketch it.

   (b) Draw a contour map for f. Remember, this is a graph of the level curves (i.e. $f(x, y) = c$ for some constant c) of f.

   (c) Use your picture from part (b) to sketch the graph of f in $\mathbb{R}^3$.

   (d) Now suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is defined by $f(x, y, z) = \sqrt{x + y}$. How do your answers to parts (a)–(c) change for this new function? In particular, can you draw the graph of f in $\mathbb{R}^3$??

4. (A Scorched Earth question!) A projectile is fired from the origin with angle of elevation α and initial speed v_0.

   (a) Assuming that air resistance is negligible and that the only force acting on the projectile is gravity, g, show that the position vector of the projectile is $\mathbf{r}(t) = (v_0 \cos\alpha)t\mathbf{i} + [(v_0 \sin\alpha)t - \frac{1}{2}gt^2]\mathbf{j}$.

   (b) Show that the maximum horizontal distance of the projectile is achieved when $\alpha = 45°$ and that in this case the range is $R = v_0^2/g$.

   (c) At what angle should the projectile be fired to achieve maximum height and what is the maximum height?

   (d) Fix the initial speed v_0 and consider the parabola $x^2 + 2Ry - R^2 = 0$, whose graph is shown in the figure. Show that the projectile can hit any target inside or on the boundary of the region bounded by the parabola and the x-axis, and that it can't hit any target outside this region.

   (e) Suppose that the gun is elevated to an angle of inclination α in order to aim at a target that is suspended at a height h directly over a point D units downrange. The target is released at the instant the gun is fired. Show that the projectile always hits the target, regardless of the value v_0, provided the projectile does not hit the ground "before" D.