ESP Workshop, Worksheet #17
Thursday November 2, 2006
AI: Eric Katerman

1. Suppose that $f(x, y)$ has continuous partial derivatives f_x and f_y at some point (x_0, y_0, z_0) where $z_0 = f(x_0, y_0)$. Last week, Dr. Durbin told us that the equation of the tangent plane to the surface $z = f(x, y)$ (that is, the graph of $f(x, y)$ in xyz-space) at the point (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Find an equation of the tangent plane to the given surface at the specified point:

(a) $z = \sqrt{4 - x^2 - 2y^2}$, $(1, -1, 1)$

(b) $z = y \ln x$, $(1, 4, 0)$

(c) $z = y \cos(x - y)$, $(2, 2, 2)$

(d) $z = e^{x^2 - y^2}$, $(1, -1, 1)$

2. A differential dz approximates the change in the value of a function $z = f(x, y)$ in terms of the changes in the variables, $dx = \Delta x$ and $dy = \Delta y$. For functions of two variables,

$$dz = f_x(x, y)dx + f_y(x, y) = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

(The three-variable case is similar.) Find the differentials of the following functions.

(a) $z = e^x \sin y$

(b) $u = r/(s + 2t)$

(c) $w = \ln \sqrt{x^2 + y^2 + z^2}$

(d) $w = xye^{xz}$

3. (An Electrical Engineering problem!) Now we will apply the above theory to a real-life situation! If R is the total resistance of the three resistors, connected in parallel, with resistances R_1, R_2, R_3, then

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

(a) Rewrite this equation to make R a function of R_1, R_2, and R_3. (All I mean is that you should invert both sides and possibly simplify to get $R = f(R_1, R_2, R_3)$.)

(b) Use your solution to part (a) to find $\partial R/\partial R_1$, i.e. the partial derivative of the function R with respect to the variable R_1.

(c) Assume for the moment that $R_3 = \infty$, i.e. there are only two resistors connected in parallel. Then R is a function of two variables, R_1 and R_2. Find the equation of a tangent plane to the surface $z = f(R_1, R_2)$ at the point $(2, 4, 4/3)$. How can you use this to estimate the total resistance if you change R_1 and R_2 a little bit?

(d) Now back to the three-variable case: if the resistances are measured in ohms as $R_1 = 25\ \Omega$, $R_2 = 40\ \Omega$, and $R_3 = 50\ \Omega$, with a possible error of 0.5% in each case, estimate the maximum error in the calculated value of R.

4. You all have seen the chain rule for real-valued functions of one variable:

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

This is a special case of the general version of the chain rule, which we'll ignore for now. (Maybe we'll look at it on Tuesday.) Another special case is when $f = f(x, y)$ is a function of two variables, where $x = x(t)$ and $y = y(t)$ are functions of a single variable t. In this case, the chain rule says that

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Remember that ∂ indicates that you're taking a partial derivative. The left-hand side of the above equation should not be confusing since f really is just a function of t. Use the chain rule to find $\frac{df}{dt}$ for the following functions:

(a) $f(x, y) = x^2y + xy^2,\ x = 2 + t^4,\ y = 1 - t^3$

(b) $f(x, y) = \sqrt{x^2 + y^2},\ x = e^{2t},\ y = e^{-2t}$

(c) $z = x\ln(x + 2y),\ x = \sin t,\ y = \cos t$

(d) Recall that $PV = 8.31T$, where P is pressure (in kilopascals), V is volume (in liters), and T is temperature (in kelvins) of a mole of an ideal gas. If the pressure is increasing at a rate of 0.05 kPa/s and the temperature is increasing at a rate of 0.15 K/s, find the rate of change of the volume when the pressure is 20 kPa and the temperature is 320 K.

Now suppose that $f = f(x, y)$ is still a function of two variables, but now the variables $x = x(s, t)$ and $y = y(s, t)$ are both functions of two variables, s and t. In this situation, the chain rule says

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Use this to find the indicated partial derivatives.

(a) $f(x, y) = x^2 + xy^3,\ x = uv^2 + v^3,\ y = u + ve^u;\ \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ when $u = 2, v = 1$

(b) $R = \ln(u^2 + v^2 + w^2),\ u = x + 2y,\ v = 2x - y,\ w = 2xy;\ \frac{\partial R}{\partial x}, \frac{\partial R}{\partial y}$ when $x = y = 1$