ESP Workshop, Worksheet #2
Tuesday September 5, 2006
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1. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function with domain the non-negative real numbers defined by an integral in the following way: for each real number c, define

$$f(c) = \int_{-c}^{c} \sin x \; dx$$

   (a) What famous function is f equal to? Draw a picture to justify your answer.

   (b) What is $f'(c)$? What about $\lim_{c\to\infty} f(c)$?

   (c) What does $\int_0^\infty \sin x \; dx$ mean? Is it convergent or divergent?

   (d) Professor Durbin claimed in class on Friday that $\lim_{c\to\infty} \int_{-c}^{c} \sin x \; dx$ exists but $\int_{-\infty}^{\infty} \sin x \; dx$ is divergent. Do you agree?[1] What is going on?

2. Jessie, the brainiac from Saved by the Bell, says, "I can't figure out the next problem and I have a test tomorrow and just in case I make the worst possible career choice ever, I need to pass!" Let's help Jessie by setting $f(x) = (\sin x)/x$, $g(x) = 1/x$, and $h(x) = f(g(x))$.

   (a) Sketch $y = h(x)$, that is, sketch $y = x\sin(1/x)$. Jessie jokes, "now that's what I call a bad hair day!"

   (b) Are f and g continuous everywhere? Jessie reminds us that continuous basically means "not broken".

   (c) Is h continuous everywhere? Does this contradict your answer to part (b)? Jessie wonders what can be said in general about the composition of continuous functions—what do you think?

   (d) Sketch $y = h'(x)$ without actually calculating $h'(x)$. Jessie remembers, "Oh yeah! $h'(x)$ is just the slope of the line tangent to the graph of $y = h(x)$ at the point $(x, h(x))$!"

   (e) (Challenge.) Jessie has a favorite function, which she calls $J(x)$, but she won't tell you what it is. All she'll tell you is that $\lim_{x\to 0} J(x) = 0$. Why couldn't you use L'Hospital's rule to evaluate

$$\lim_{x\to 0} \frac{J(x)}{h(x)}?$$

[1] Remember, he also claimed that "1/3 of all integrals converge," so it may be okay to disagree with him from time to time...

3. Let $f(x) = \lfloor x \rfloor$. This is called the "floor" function, and it returns the largest integer less than or equal to x. Also, let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be defined by $g(x) = (f(x))!$.

   (a) Draw the graph of $y = f(x)$ for x in the range $-3 \leq x \leq 3$. What is $f(-1.5)$? What about $f(1.9999)$? And $f(1.\overline{9})$?

   (b) Draw the graph of $y = g(x)$ for x in the range $0 \leq x \leq 4$.

   Let's define a new function, called $\Gamma(x)$, like this:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \qquad x > 0.$$

   (c) Use integration by parts to prove that $\Gamma(x+1) = x\Gamma(x)$.

   (d) Show that $\Gamma(1) = 1$. Conclude that $\Gamma(n) = (n-1)!$ for all natural numbers n.

   The gamma function provides an example of a function (continuous on the positive real numbers) which **interpolates** the values of $n!$ for natural numbers n.

   (e) (Challenge.) Show that $\Gamma(1/2) = \sqrt{\pi}$.[2]

4. (a) Evaluate $\int_1^\infty \frac{1}{x}\, dx$, draw the graph of $y = 1/x$ and indicate (graphically) what this integral is measuring.

   (b) For what values of p does $\lim_{t\to\infty} t^{1-p}$ converge? Diverge?

   (c) Suppose $p < 1$. What is

$$\int_1^\infty \frac{1}{x^p}\, dx?$$

   (d) Now suppose $p > 1$. What is $\int_1^\infty x^{-p}\, dx$? What about

$$\int_1^\infty 2\pi x^{1-p}\, dx$$

   and what does it measure? (Hint: what's Eric's least-favorite hole in putt-putt?) Draw a picture!

5. There are some ESP students all standing in a circle.

   (a) First suppose that every guy is next to two girls and every girl is next to two guys (i.e. the circle alternates). If there are 25 girls, how many guys must there be?

   (b) (Challenge.) Those same students rearrange themselves in some crazy order (not necessarily alternating). Prove that both neighbors of at least one student are guys.

[2] This is probably really hard or impossible at this point. We'll come back to it after we have seen some multi-variable calculus...