ESP Workshop, Worksheet #20
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1. Recall the second derivative test to find local minima/maxima of multivariable functions: Suppose that $f_x(a,b) = 0$ and $f_y(a,b) = 0$ (that is, (a,b) is a critical point of f). Let
$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$
   (a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
   (b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
   (c) If $D < 0$, then $(a,b,f(a,b)) \in \mathbb{R}^3$ is a saddle point of the graph of f.

   Let's use this to find the local minimum and maximum values and saddle points of the following functions:

   (a) $f(x,y) = 9 - 2x + 4y - x^2 - 4y^2$
   (b) $f(x,y) = x^4 + y^4 - 4xy + 2$
   (c) $f(x,y) = 1 + 2xy - x^2 - y^2$

2. (Optional; requires technology) If you have a graphing calculator that graphs three-dimensional things, plot the functions from question 1 and identify local minima, maxima, and saddle points.

3. Evaluate the following double integrals; note that for some of them, it will help to first identify the integral as the volume of a solid. Draw a picture of each solid in xyz-space.

   (a) $\iint_R 3\ dA,\ R = \{(x,y)| -2 \le x \le 2, 1 \le y \le 6\}$
   (b) $\iint_R (5-x)\ dA,\ R = \{(x,y)|0 \le x \le 5,\ 0 \le y \le 3\}$
   (c) $\iint_R \sqrt{9-y^2}\ dA,\ R = [0,4] \times [0,2]$

4. Recall that the average value of a function f of one variable is
$$\frac{1}{b-a}\int_a^b f(x)dx$$
   That is, we divide the integral by the length of the interval we're integrating along. Similarly, the average value of a function f of two variables defined on a region R is
$$\frac{1}{A(R)}\iint_R f(x,y)dA$$
   where $A(R)$ is the area of R.

Suppose that the temperature in Texas yesterday at noon was

$$T(x, y) = 60 + y^2 - xy$$

where Austin is the origin and 1 unit in the positive x and y directions represent 100 miles east and north, respectively. Approximate the average temperature in Texas by finding the average value for T in the region $R = [-6, 3] \times [-3, 3]$. How well does R approximate Texas? Can you define a better region R' for Texas? How do the averages compare for the different regions?

5. First try to evaluate the integral in the way it is presented, and then reverse the order of integration and try it that way. Don't forget to change the limits of integration (drawing a picture will help)! Which integral was easier?

   (a) $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$

   (b) $\int_0^9 \int_0^{\sqrt{x}} y \cos(x^2)\, dy\, dx$

   (c) $\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1 + \cos^2 x}\, dx\, dy$

6. (Challenge!) The Riemann zeta function is one of the most famous functions in all of mathematics, and it is the subject of one of the unsolved million dollar prize problems. It is defined as an infinite series:

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

   Now, we can finally prove that $\zeta(2) = \pi^2/6$, as promised! Here we go!

   (a) The double integral $\int_0^1 \int_0^1 \frac{1}{1-xy} dx\, dy$ is an improper integral and could be defined as the limit of double integrals over the rectangle $[0, t] \times [0, t]$ as $t \to 1^-$. But if we expand the integrand as a geometric series, we can express the integral as the sum of an infinite series. Show that

$$\int_0^1 \int_0^1 \frac{1}{1 - xy} dx\, dy = \zeta(2)$$

   (b) Evaluate that double integral by first making the substitution

$$x = \frac{u - v}{\sqrt{2}} \qquad y = \frac{u + v}{\sqrt{2}}$$

   (Note: this substitution gives a rotation about the origin through the angle $\pi/4$.) Sketch the corresponding region in the uv-plane. Hint: if, in evaluating the integral, you encounter either of the expressions

$$\frac{1 - \sin\theta}{\cos\theta} \quad \text{or} \quad \frac{\cos\theta}{1 + \sin\theta}$$

   you might like to use the identity $\cos\theta = \sin((\pi/2) - \theta)$ and the corresponding identity for $\sin\theta$.