ESP Workshop, Worksheet #21
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1. Finally... some applications of all this abstract nonsense we've been learning over the past few months! Let's make some real-world computations!

   (a) Electric charge is distributed over the rectangle $1 \leq x \leq 3,\ 0 \leq y \leq 2$ so that the charge density at (x, y) is
   $$\sigma(x, y) 2xy + y^2$$
   (measured in coulombs per square meter). Find the total charge on the rectangle.

   (b) Find the mass and center of mass of the thin plate (or "lamina") that occupies the region bounded by $y = e^x, y = 0, x = 0$, and $x = 1$ and has density function $\rho(x, y) = y$.

   (c) Find the area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

   (d) Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

2. It's that time again... time to do some of Professor Durbin's old exam problems.

   (a) Suppose that f has continuous first and second partial derivatives for $x^2 + y^2 < 9$ and that $f_x(1, 2) = 0,\ f_y(1, 2) = 0$, and $f_{xx}(1, 2) < 0$. Let $d = f_{xx}(1, 2) f_{yy}(1, 2) - [f_{xy}(1, 2)]^2$. What can you conclude about the local extrema of f in each of the following cases?

      i. $d < 0$

      ii. $d = 0$

      iii. $d > 0$

   (b) Write (but do not evaluate) a double integral in polar coordinates that will give the volume of the solid bounded above by the spherical surface $x^2 + y^2 + z^2 = 9$ and below by the plane $z = 2$.

   (c) Find the minimum distance from the point (0,1) to the parabola $y = x^2$.

   (d) Locate and classify all extreme points of $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$.

   (e) Change the order of integration:
   $$\int_0^2 \int_{x^2}^{2x} f(x, y) dy dx$$
   Now let $f(x, y) = 3x + 5xy$ and evaluate the double integral both ways. Did you get the same answer?

(f) Assume $f(x, y) = 3x + y$ on the region $R = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 2\}$. Also assume that $P_1 = \{0, 1/2, 2\}$, $P_2 = \{0, 1, 2\}$, and $P = P_1 \times P_2$. Find the lower approximating sum $L(P)$. How does this compare with the actual value of the double integral on R?

(g) Evaluate the following double integral with Ω the triangle formed by the x-axis and the lines $y = x$ and $x = 3$.

$$\iint_\Omega e^{x+y} dy dx$$

Check your answer by comparing it with $\iint_\Omega e^{x+y} dx dy$.

3. The figure at the bottom of the page shows the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$. Find the area of this surface.

4. (Math Challenge!) Evaluate the integral

$$\int_0^1 \int_0^1 e^{\max\{x^2, y^2\}} dy dx$$

where $\max\{x^2, y^2\}$ means the larger of the numbers x^2 and y^2.

5. (Physics Challenge!) A lamina has constant density ρ and takes the shape of a disk with center the origin and radius R. Recall that Newton's Law of Gravitation

$$\mathbf{F} = -\frac{GMm}{r^3}\mathbf{r}$$

(where $\mathbf{F}$ is the gravitational force on the body of mass m, M is the mass of the lamina, G is the gravitational constant, and $\mathbf{r}$ is the position of the smaller body).

(a) Show that the magnitude of the force of attraction that the lamina exerts on a body with mass m located at the point $(0, 0, d)$ on the positive z-axis is

$$F = 2\pi G m \rho d \left(\frac{1}{d} - \frac{1}{\sqrt{R^2 + d^2}} \right)$$

[Hint: Use polar coordinates, and divide the disk into "polar subrectangles" and first compute the vertical component of the force exerted by the polar subrectangle R_{ij}.]

(b) Show that the magnitude of the force of attraction of a lamina with density ρ that occupies an entire plane on an object with mass m located at a distance d from the plane is $F = 2\pi G m \rho$. (Notice that this expression does not depend on d!)