

ESP Workshop, Worksheet #22
Thursday November 30, 2006
AI: Eric Katerman

One of the most important skills to learn from this course—and one that will certainly be tested on either the next exam or the final—is to be able to correctly **identify limits of integration for double and triple integrals**. Professor Durbin described this a bit yesterday for triple integrals, and he admitted to me that he didn't present it the way he should have. Let's learn the easy way to identify regions of integration. For each of the following integrals, follow these steps:

1. If a region D is not given to you explicitly, write down the region in set notation: eg. $D = \{(x, y) : -1 \leq x \leq 1, -x \leq y \leq x^2\}$
2. Draw the region in the xy -plane (for double integrals) or in xyz -space (for triple integrals).
3. Work from the “inside-out” to check your picture: eg. if you were integrating

$$\iiint_D f(x, y, z) dx dy dz$$

, draw the projection of your picture from part (b) onto the yz -plane, and then onto the z -axis.

4. Write down all possible rearrangements of the integral, with the proper bounds; for double integrals, there are two possibilities ($dx dy$ and $dy dx$), and for triple integrals, there are six (which ones?). To do this, take your picture from part (b), and follow the same steps as in part (c). For example, if now we were doing $\iiint_D f(x, y, z) dz dx dy$, we would first project onto the xy -plane and then onto the y -axis.
5. (Optional.) Evaluate a few of the rearrangements to check your answers.

See? It's a simple four-step process. (NB: you won't always be able to do each of these steps—the obstruction has to do with type I vs. type II regions and so forth. For more on that, see section 16.7 in the text.) Now let's practice!

1. $\int_0^1 \int_{3y}^3 e^{x^2} dx dy$
2. $\int_0^9 \int_0^{\sqrt{x}} y \cos(x^2) dy dx$
3. $D = \{(x, y, z) : x^2 + z^2 = 4, y = 0, y = 6\}$
4. $\int_0^1 \int_y^1 \int_0^y (xyz) dz dx dy$
5. $\int_0^1 \int_0^{1-x^2} \int_0^{1-x} (xy + yz) dy dz dx$

6. (Challenge!) In this problem, we will find the volume enclosed by a hypersphere in n -dimensional space! Ooooooh!

(a) Use a double integral and trig substitution, together with the formula:

$$\int \cos^2 u \, du = \frac{1}{2}u + \frac{1}{4} \sin 2u + C$$

to find the area enclosed by a circle of radius r .

(b) Use a triple integral and a trig substitution to find the volume inside a sphere with radius r .

(c) Use a quadruple integral to find the hypervolume enclosed by the hypersphere $x^2 + y^2 + z^2 + w^2 = r^2$ in \mathbb{R}^4 . Hint: use trig substitution and the following reduction formulas:

$$\begin{aligned} \int \sin^n x \, dx &= -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \\ \int \cos^n x \, dx &= \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx \end{aligned}$$

(d) Use an n -tuple integral to find the volume enclosed by a hypersurface of radius r in n -dimensional space \mathbb{R}^n . Hint: the formulas are different for n even and n odd.