ESP Workshop, Worksheet #3
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1. Consider the sequence $\{1, -1/2, 1/3, -1/4, \ldots\}$.
   (a) Find a formula for the general term a_n of the sequence.
   (b) Sketch the sequence both as points on a number line and as a graph of a function whose domain is the set of positive integers.
   (c) Find the limit of this sequence as $n \to \infty$.
   (d) Repeat parts (a) – (c) with the sequence $\{2/9, 6/27, 24/81, 120/243, \ldots\}$

2. Now we use the Squeeze theorem...
   (a) Derive $\sin(\pi/4) = \sqrt{2}/2$ using an argument involving isosceles triangles. Also show that $\sin(\pi/6) = 1/2$ using equilateral triangles.
   (b) Consider the sequence $a_n = \sin(n\pi/6), n \geq 0$. Write out the first thirteen terms. What is a_{100}?
   (c) Does this sequence converge? What about the sequence $b_n = a_n/(\ln n)$? Justify your answers, and sketch a_n and b_n as graphs (with domain the positive integers).
   (d) Find
   $$\lim_{n\to\infty} \frac{n + 2\cos n}{n}$$
   and sketch the graph to see what's going on.

3. Let $f(x) = \lfloor x \rfloor$. This is called the "floor" function, and it returns the largest integer less than or equal to x. Also, let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be defined by $g(x) = (f(x))!$. This function g *extends* the factorial function, which is only defined for non-negative integers: $n! : \mathbb{Z}_{\geq 0} \to \mathbb{Z},\ n! = n(n-1)\cdots 2 \cdot 1$.
   (a) What's the limit of the sequence $\{1.9, 1.99, 1.999, 1.9999, \ldots\}$?
   (b) Draw the graph of $y = f(x)$ for x in the range $-3 \leq x \leq 3$. What is $f(-1.5)$? What about $f(1.9999)$? And $f(1.\overline{9})$?
   (c) Draw the graph of $y = g(x)$ for x in the range $0 \leq x \leq 4$.
   Let's define a new function, called $\Gamma(x)$, like this:
   $$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \qquad x > 0.$$
   (d) Use integration by parts to prove that $\Gamma(x+1) = x\Gamma(x)$.

(e) Show that $\Gamma(1) = 1$. Conclude that $\Gamma(n) = (n-1)!$ for all natural numbers n.

The gamma function is very special: not only does it *extend* the factorial function (like f), but it is actually *differentiable* on the positive real line! Can you think of a continuous function that extends the sequence $a_n = (-1)^n$ to $\mathbb{R}$? What about a differentiable one?

4. To construct the **snowflake curve**, start with an equilateral triangle with sides of length 1. Step 1 in the construction is to divide each side into three equal parts, construct an equilateral triangle on the middle part, and then delete the middle part. Step 2 is to repeat Step 1 for each side of the resulting polygon. This process is repeated at each succeeding step. The snowflake curve is the curve that results from repeating this process indefinitely.

   (a) Draw the first few stages of this construction.

   (b) Let s_n, l_n, and p_n represent the number of sides, the length of a side, and the total length of the nth approximating curve (the curve obtained after Step n of the construction), respectively. Find formulas for s_n, l_n, and p_n.

   (c) Show that $p_n \to \infty$ as $n \to \infty$, so the snowflake curve has infinite perimeter!

   (d) Give an easy argument to show that the snowflake curve have finite area. Can you compute its area explicitly?

5. We say that the set of all real numbers $\mathbb{R}$ is **complete** because every non-empty set of real numbers with an upper bound *in* $\mathbb{R}$ has a least upper bound *in* $\mathbb{R}$, so there are no "holes" in $\mathbb{R}$. We will show that the set of all rational numbers

$$\mathbb{Q} = \left\{ \frac{r}{s} : r, s \text{ are integers} \right\}$$

   does not have this property. Here's the strategy: we will find a real number x that is not rational, and then we will construct a subset of $\mathbb{Q}$ whose least upper bound *in* $\mathbb{R}$ is x.

   (a) Suppose that $\sqrt{2}$ is rational, i.e. suppose $\sqrt{2} = a/b$ for some integers a, b. Also suppose that a/b IS IN LOWEST TERMS. What does $2b^2$ equal?

   (b) Is a^2 odd or even? (If you can't tell, take another look at part (a).)

   (c) Is b^2 odd or even? Why does this prove that $\sqrt{2}$ is not rational?

   (d) Let S be the set of all rational numbers less than $\sqrt{2}$, i.e.

$$S = \{x \in \mathbb{Q} : x < \sqrt{2}\}$$

   Use the fact that between any two distinct real numbers $x, y \in \mathbb{R}$, $x \neq y$, there exists a rational number $r/s \in \mathbb{Q}$ (i.e. $x < r/s < y$) to show that S does not have a least upper bound in $\mathbb{Q}$.

   (e) (Challenge.) Can you prove the fact we used in the last part?