ESP Workshop, Worksheet #4
Tuesday September 12, 2006
AI: Eric Katerman

1. (a) Define the two terms *bounded* and *monotonic* as they apply to sequences and say what you can about a sequence that is both bounded and monotonic.

   For the following sequences, determine whether each is bounded, monotonic, and convergent or divergent.

   (b) $a_n = \frac{n!}{n^n}$

   (c) $a_n = \frac{1}{n}$

2. Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

3. Evaluate:

   (a) $\sum_{i=1}^{100} 2$

   (b) $\sum_{i=1}^{100} \frac{1}{i+3} - \frac{1}{i+4}$

   (c) $\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$

4. In class we learned about geometric series. They are of the form

$$a + ar + ar^2 + ar^3 + \ldots = \sum_{i=0}^{\infty} ar^i$$

   (a) Let s_n be the n-th partial sum of this series.

$$s_n = a + ar + ar^2 + \ldots + ar^n \qquad \text{(there are n+1 terms here)}$$

   Calculate $s_n - rs_n$.

   (b) Using this, find a "closed" formula for s_n.

   (c) Compute $\lim_{n\to\infty} s_n$. How does this limit depend on r?

5. (a) Converge or Diverge?

$$\frac{\sin(\theta)}{2} + \frac{\sin^2(\theta)}{4} + \frac{\sin^3(\theta)}{8} + \frac{\sin^4(\theta)}{16} + \ldots$$

   If it converges can you tell what it converges to?

(b) Converge or Diverge?

$$\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$$

If it converges can you tell what it converges to?

6. If a and b are digits, show (using arguments involving geometric series like those we saw in class):
   (a) $0.\overline{a} = \frac{a}{9}$
   (b) $0.\overline{ab} = \frac{10a+b}{99}$
   (c) $.\overline{9} = 1$
   (d) Write $3.7\overline{2}$ as a fraction.

7. (a) Evaluate $\int_1^\infty \frac{1}{x}\, dx$, draw the graph of $y = 1/x$ and indicate (graphically) what this integral is measuring.
   (b) For what values of p does $\lim_{t\to\infty} t^{1-p}$ converge? Diverge?
   (c) Suppose $p < 1$. What is

$$\int_1^\infty \frac{1}{x^p}\, dx?$$

   (d) Now suppose $p > 1$. What is $\int_1^\infty x^{-p}\, dx$? What about

$$\int_1^\infty 2\pi x^{1-p}\, dx$$

   and what does it measure? (Hint: what's Eric's least-favorite hole in putt-putt?) Draw a picture!
   (e) What does the integral test for convergence say? Use it to decide for which p the series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges.

8. Compute the area of the snowflake curve from the previous worksheet, and, if you haven't done so yet, show that it has infinitely long perimeter.

9. (Challenge.) Find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$$

where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.