ESP Workshop, Worksheet #5
Thursday September 14, 2006
AI: Eric Katerman

1. Eric is feeling really generous today! He tells you that these problems were on one of Professor Durbin's old M408D exams!

   (a) Determine the convergence or divergence of

   $$\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$$

   (b) Determine the convergence or divergence of

   $$\sum_{n=1}^{\infty} \frac{1}{5^n + 3}$$

   (c) Suppose the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges to the sum S and that $0 < a_{n+1} \leq a_n$ for all n. What can you say about the difference between S and $\sum_{n=1}^{N}(-1)^n a_n$? (Eric says: "Maybe it would help to draw a picture like Professor Durbin drew on the chalkboard yesterday...")

2. Use the limit comparison test to determine whether

   $$\sum_{n=1}^{\infty} \frac{2n^2 + 5}{2\sqrt{n + n^7}}$$

   converges or diverges.

3. It turns out that Eric actually needs a favor, and that's the only reason he was being nice! Eric wants to pass his Math Subject Test GRE so he teach calculus at UT, and he needs your help with some problems from his practice book!

   (a) Which of the following series converge, and which diverge?

   $$\text{(i)} \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3 + n} \qquad \text{(ii)} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}} \qquad \text{(iii)} \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

   (b) Consider the sequence $\{a_n\}$ whose terms are given by the formula

   $$a_n = \frac{(\cos n\pi)(\sin^2 n)}{\sqrt{n}}$$

   for each integer $n \geq 1$. If this sequence converges, what is the limit?

(c) Which of the following series converge?

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^4(\arctan n)}{n\sqrt{n}}$ (ii) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ (iii) $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)^3}{5(n+2)(n+3)(n+4)}$

(d) Which of the following statements are true?

i. If $a_n \geq 0$ for every n, then:

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} \sqrt{a_n} \text{ converges}$$

ii. If $a_n \geq 0$ for every n, then:

$$\sum_{n=1}^{\infty} n a_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

iii. If $a_n \geq 0$ and $a_{n+1} \leq a_n$ for every n, then:

$$\sum_{n=1}^{\infty} a_n^2 \text{ converges} \Rightarrow \sum_{n=1}^{\infty} (-1)^n a_n \text{ converges}$$

(e) (Challenge.) If $|x| < 1$, then compute

$$\sum_{n=1}^{\infty} n x^{2n}$$

4. (Challenge.) Find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$$

where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.

5. (This one is for all you engineering majors...) Suppose you have a large supply of books, all the same size, and you stack them at the edge of a table, with each book extending farther beyond the edge of the table than the one beneath it. Show that it is possible to do this so that the top book extends entirely beyond the table. In fact, show that the top book can extend any distance at all beyond the edge of the of the table if the stack is high enough (!). Use the following method of stacking: the top book extends half its length beyond the second book. The second book extends a quarter of its length beyond the third. The third extends one-sixth of its length beyond the fourth, and so on. HINT: consider centers of mass.