ESP Workshop, Worksheet #6

Tuesday September 19, 2006

AI: Eric Katerman

1. Identify in each problem below which test you would use to determine convergence. Do not do the work, just identify the most effective test.

   a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)!}$

   d) $\displaystyle\sum_{n=1}^{\infty} \frac{n3^n}{4^{n-1}}$ e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ f) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\sqrt{n^2-1}}$

2. Let x be a real number. Use the ratio test to determine the values of x for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n^3}$ converges. Try the same for $\sum_{n=1}^{\infty} \frac{x^n}{n!}$. (For future reference, this set of values is commonly called the "radius of convergence".)

3. Test each series for convergence or divergence. For convergent alternating series, classify the convergence as absolute or conditional.

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n^2+n+1}$ b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{1+8n}\right)^n$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{3^n}$

   d) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{2n-1}{n+4}\right)$ e) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$ f) $\displaystyle\sum_{n=1}^{\infty} e^{-n^2}$

   g) $\displaystyle\sum_{n=1}^{\infty} \sin^3\left(\frac{1}{n}\right)$ h) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3-1}$ i) $\displaystyle\sum_{n=1}^{\infty} \frac{n3^n}{4^{n-1}}$

   (Hint for part (g): First make a graphical argument to show that for all positive x, $x \geq \sin x$. What is the derivative of $\sin x$ at $x = 0$?)

4. Define a function $f(x)$ by the following series:

$$f(x) = \sum_{n=0}^{\infty} x^n$$

   (a) This is a power series. What is the "center" of this power series? (That is, what is the center of the interval of convergence?)
   (b) What is the interval of convergence of this function, i.e. for what values of $x \in \mathbb{R}$ does the series converge?
   (c) Find a simpler expression for $f(x)$. Find its derivative, and then graph $y = f(x)$.
   (d) What is the value of $f(-1)$? Use both your simpler expression from part (b) and the original series defintion to evaluate it. What happened??

5. For each power series, find the radius of convergence and determine if it converges at each of the endpoints of the interval of convergence.

   a) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{\sqrt{n}}$ b) $\sum_{n=1}^{\infty} x^n n^n$ c) $\sum_{n=1}^{\infty} \frac{6^n}{n^2} x^n$ d) $\sum_{n=1}^{\infty} \frac{(2x)^n}{n^n}$

6. Geometric and power series are often confused. Power series are of the form $\sum c_n x^n$, where the c_n are possibly different constants, x an unknown constant.[1] Geometric series are of the form $\sum ar^n$ where r and a are constants. For each series below, determine if it is geometric, power, both or neither.

   a) $1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+...$ b) $1+1.1+1.21+1.331+1.4641+...$ c) $(\frac{1}{3})^2+(\frac{1}{3})^4+(\frac{1}{3})^6+...$

   d) $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+...$ e) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...$ f) $\frac{\ln x}{2}+\frac{\ln x}{3}+\frac{\ln x}{4}+\frac{\ln x}{5}+...$

7. We say that a plane figure has "n-degree rotational symmetry" if it looks the same after rotating it by n degrees around some point. For example, a square has 90-degree rotational symmetry. Now you're told that a certain plane figure has 19-degree rotational symmetry. Prove that it also has 1-degree rotational symmetry. Can you sketch such a figure?

[1] We sometimes call x a variable, especially when we're thinking of the power series as defining a function $f(x)$. However, for each value of x, we may think of x as a constant in the power series since it does not depend on the index n.