ESP Workshop, Worksheet #8
Thursday September 28, 2006
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1. You know a power series representation for $\frac{1}{1-x}$. Recalling this, use term by term integration or differentiation, or direct substitution to find a power series representation for the following functions. State the inverval of convergence for each power series.

   (a) $\frac{1}{(1-x)^2}$ (b) $\frac{1}{1+x}$

   (c) $\frac{1}{1+x^2}$ (d) $\tan^{-1}(x)$

   (e) $\ln\left(\frac{1+x}{1-x}\right)$ (f) $(\ln(1-x)-1)(1-x)$

   Hint for (f): Take derivatives until you recognize what to do.

2. Taylor series may be found for these functions by a long method or a short method. Describe both methods, then use the short one.

   (a) $f(x) = \frac{\sin x}{x}$ (b) $f(x) = x^3 e^x$ (c) $f(x) = \ln(x^2+1)$

3. Recall that a power series centered around $x = a$ has the general form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

   for some coefficients $c_n \in \mathbb{R}$. For each of the following power series, find the center a, the coefficients c_n, the radius of convergence R, and the interval of convergence. For the last part, don't forget to check the endpoints!

   (a) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$ (b) $\sum_{n=0}^{\infty} \frac{(x+3)^n}{5^n}$

   (c) $\sum_{n=0}^{\infty} (-1)^n \frac{(x+4)^n}{n+2}$ (d) $\sum_{n=0}^{\infty} n!(x-1)^n$

4. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Find the intervals of convergence for f, f', and f''.

5. (a) Starting with the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, find the sum of the series

$$\sum_{n=1}^{\infty} n x^{n-1} \qquad |x| < 1$$

   (b) Find the sum of each of the following series.

$$(i) \sum_{n=1}^{\infty} n x^n, \; |x| < 1 \qquad (ii) \sum_{n=1}^{\infty} \frac{n}{2^n}$$

   (c) Find the sum of each of the following series.

$$(i) \sum_{n=2}^{\infty} n(n-1)x^n, \; |x| < 1 \qquad (ii) \sum_{n=2}^{\infty} \frac{n^2 - n}{2^n} \qquad (iii) \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

6. In class the other day, Professor Durbin observed that the Taylor series centered around $x = 0$ of $f(x) = \sin x$ (an odd function) had no terms with even powers of x. Can you prove that this is true for odd functions in general? That is, can you show that if f is an odd function then the Taylor series for f at zero will contain only terms with odd powers?

7. (a) Show that the function defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   is not equal to its Maclaurin series.

   (b) Graph the functions $e^x, -1/x^2$, and e^{-1/x^2}. Use the third graph to comment on the behavior near the origin of $f(x)$ from part (a).

8. (Challenge.) Find the sum of the series

$$\sum_{n=2}^{\infty} \ln \left(1 - \frac{1}{n^2}\right)$$