

ESP Workshop, Worksheet #9
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1. Find the power series expansion for the given function about the given a :

$$(a) f(x) = \frac{1}{x^2 - 2x - 3}, \quad a = 0 \qquad (b) f(x) = \frac{1}{x^2 - 2x + 2}, \quad a = 1$$

2. Recall that Taylor's theorem tells us that the coefficients of a power series for a function are given in terms of the derivatives of the function:

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n, \quad c_n = \frac{f^{(n)}(a)}{n!}$$

We've used this to find power series for functions, but you can use it *backwards*, too! Check it out:

- (a) for $f(x) = \cos(x^{5/2})$, find $f^{(15)}(0)$, $f^{(27)}(0)$, $f^{(40)}(0)$.
(b) for $g(x) = (2 + x)^{-1}$, find $g^{(6)}(7/2)$.
(c) for $h(x) = (2 - x)^{3/2}$, find $f^{(11)}(4)$.
3. For any real number k , we can define binomial coefficients like this:

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

for all $n \geq 1$ and

$$\binom{k}{n} = 0$$

for $n = 0$. Use this and the binomial series to expand the following function as a power series (and state the radius of convergence):

$$f(x) = \sqrt{1 + x^2}$$

As in question 2, use your answer to find $f^{(10)}(0)$.

4. Let's keep going backwards! Find each of the following sums; for example, if I gave you the sum $1 + x + x^2 + x^3 + \dots$, you would answer $\frac{1}{1-x}$.

(a) $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$

(b) $1 - x^3 + x^6 - x^9 + \dots$

(c) $\frac{x^2}{2} - \frac{x^3}{3 \cdot 2} + \frac{x^4}{4 \cdot 3} - \frac{x^5}{5 \cdot 4} + \dots, |x| < 1$

5. Let's practice working with the dot product! Use its definition

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2$$

(where $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$) to prove the properties that Professor Durbin mentioned yesterday:

- (a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- (b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- (c) $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v}$ where c is a scalar (i.e. a number, not a vector)
- (d) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

The proofs in the 3-dimensional case are exactly the same—if you want to, you should work through them!

6. (A Chemistry problem!) A molecule of methane, CH_4 , is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The *bond angle* is the angle formed by the H–C–H combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about 109.5 degrees. [*Hint*: Take the vertices of the tetrahedron to be the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$ as shown on the board. Then the centroid is $(1/2, 1/2, 1/2)$.]
7. Find the volume of the solid that lies inside both of the spheres

$$x^2 + y^2 + z^2 + 4x - 2y + 4z + 5 = 0$$

and

$$x^2 + y^2 + z^2 = 4$$

8. (An Aerospace Engineering problem!) Suppose the three coordinate planes are all mirrored and a light ray given by the vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ first strikes the xz -plane, as shown on the blackboard. Use the fact that the angle of incidence equals the angle of reflection (what's the name of that law?) to show that the direction of the reflected ray is given by $\mathbf{b} = \langle a_1, -a_2, a_3 \rangle$. Deduce that, after being reflected by all three mutually perpendicular mirrors, the resulting ray is parallel to the initial ray.

(Challenge.) Can you see how to use this principle, together with laser beams and an array of these “corner mirrors” on the moon, to calculate precisely the distance from the Earth to the Moon??