Short History of Numbers

1) When Mother still dwelt in the trees and counted fingers, bananas, and children, she invented the Natural Numbers

$$\{\mathbb{N}, +, \times, \leq\} \qquad\qquad \mathbb{N} \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}\ .$$

This enabled her to solve equations like

$$a + x = b\ , \qquad\qquad a < b\ .$$

2) When she started a small business trading bananas, she experienced the need to solve equations like

$$5 + x = 3\ .$$

This cannot be done in $\mathbb{N}$, so she extended the number system by adding zero and

negative integers, arriving at the larger number system

$$\{\mathbb{Z}, +, \times, \leq\} \qquad \mathbb{Z} \stackrel{\text{def}}{=} \{0, 1, -1, 2, -2, \ldots\}\ .$$

of Integers.

3) When she had a birthday party with 5 kids but bananas only for 3 birthday cakes, she experienced the need to solve equations like

$$5 \times x = 3\ .$$

Namely, she knew that she must give every kid an equal amount x of cake lest she invite a mutiny. She found it easier to extend the number system again, to the Rational Numbers

$$\{\mathbb{Q}, +, \times, \leq\} \quad \mathbb{Q} \stackrel{\text{def}}{=} \{m/n :\ m, n \in \mathbb{Z}, n \neq 0\}.$$

4) After the birthday party every piece of cloth in her hut was soiled, so she washed all. Since it rained, she decided to hang up the laudry to dry inside her hut. She needed a clothesline the length of the diagonal of her square hut. Having spent her free time on a bit of geometry just for fun, she knew that the length x of the clothesline, measured in [hutwidths] satisfied the Pythagorean equality

$$x^2 = 2\ .$$

She saw that $\mathbb{Q}$ does not contain any number x with square 2, so what did she do?

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to the Real Numbers

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How did she do that?

She took for $\mathbb{R}$ all the holes in $\mathbb{Q}$ and defined $+, \times, \leq$ suitably. Then she could send the oldest to the hardware store to buy a rope of length $\sqrt{2}$ [hutwidths].

5) Being successful is fun, she was evidently good in mathematics, so she continued on, and asked herself about the equation

$$x^2 = -1 \ .$$

There is no real number x with square -1, since the square of any real is positive. so what did she do?

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She put

$$\mathbb{C} \overset{\text{def}}{=} \{(a,b) : \ a,b \in \mathbb{R}\}$$

$$= \{a + ib : \ a,b \in \mathbb{R}\} \qquad 1 = (1,0), i = (0,1)$$

and defined addition and multiplication by

$$(a,b) + (a',b') \ \overset{\text{def}}{=} \ (a+a', b+b') \quad \textbf{and}$$
$$(a,b) \times (a',b') \ \overset{\text{def}}{=} \ (aa' - bb', ab' + a'b) \ .$$

There is no way to define on $\mathbb{C}$ an order that collaborates with addition and multiplication as the order on $\mathbb{R}$ does. That is a drawback. But there is at least an

absolute value

$$|(a,b)| \stackrel{\text{def}}{=} \sqrt{a^2+b^2}\ ,$$

which makes analysis on $\mathbb{C}$ possible (and quite amazing!). Not only does the equation

$$z^2+1=0\ , \qquad z \in \mathbb{C}$$

have a solution, to wit $i \stackrel{\text{def}}{=} (0,1)$, every complex polynomial

$$p(z) = \sum_{n=0}^{N} a_n z^n \qquad a_n \in \mathbb{C}$$

has a root, in fact is a product of linear polynomials.
You can talk about limits of sequences in $\mathbb{C}$, and about continuity and differentiabilty of functions $f : \mathbb{C} \to \mathbb{C}$, and about

power series. In fact, every once differentiable function is locally a power series.

Example: **Consider the real power series**

$$e^x \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!}\,, \qquad\qquad x \in \mathbb{R}\ .$$

It converges for every x**,**

satisfies $e^{x+x'} = e^x \times e^{x'}$**, and** $\dfrac{de^x}{dx} = e^x$**.**

The complex exponential series

$$e^z \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{z^n}{n!}\,, \qquad\qquad z \in \mathbb{C}$$

has the same properties:

it converges for every z **(same proof as in** $\mathbb{R}$**),**

satisfies $e^{z+z'} = e^z \times e^{z'}$ **(same algebra as in** $\mathbb{R}$**),**

and $\dfrac{de^z}{dz} = e^z$ **(same proof as in** $\mathbb{R}$**).**
Let $z = ix = (0, x)$**,** $x \in \mathbb{R}$**. Then**

$$\begin{aligned} e^{ix} &= 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \cdots \\ &+ i\Big(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} \mp \cdots\Big) \\ &= \cos x + i \sin x \ . \end{aligned}$$

This formula,

$$e^{ix} = \cos x + i \sin x$$

is called Euler's formula and will assist us greatly in solving HCCSOLODE with strictly negative discriminant. [I was baffled when I first met it.]