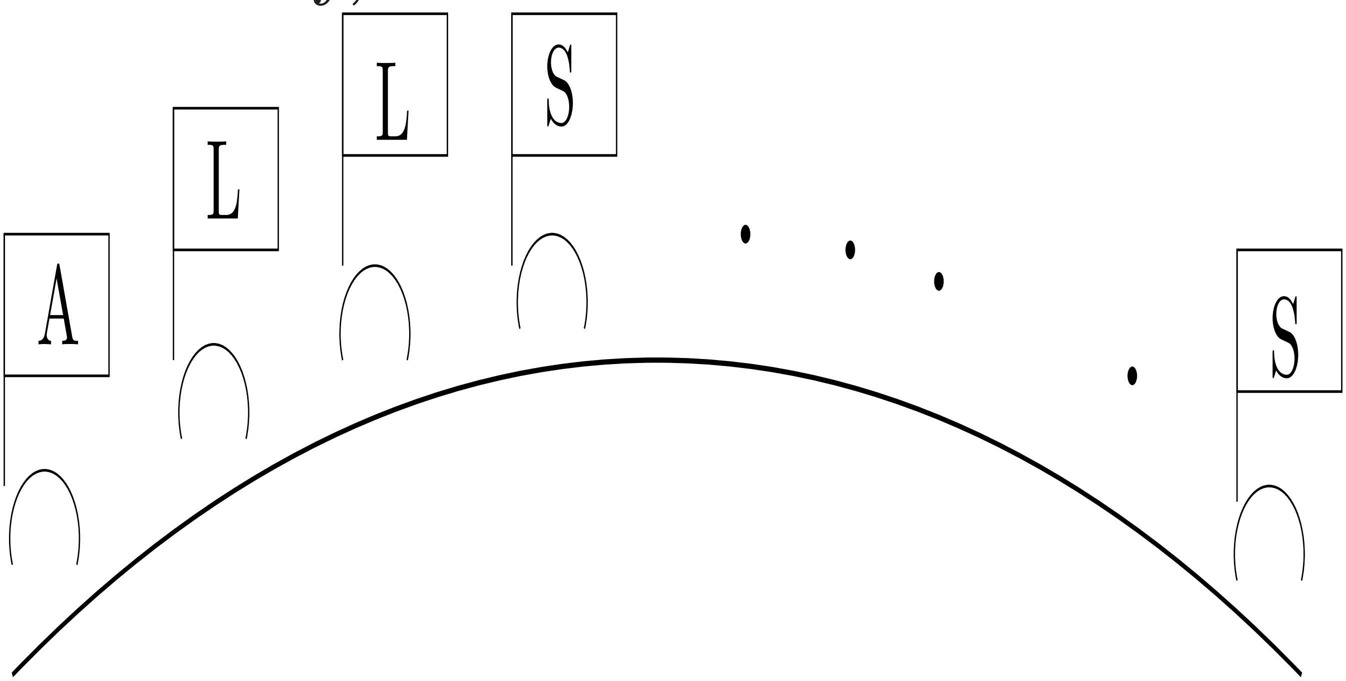
Lecture 2

Review: **Principle 0, Principle 1, and**
Theorem: **A set of size n has**

$$\binom{n}{k} \stackrel{\text{def}}{=} \frac{n!}{k!(n-k)!}$$

subsets of size k if $k \leq n$, none if $k > n$.

Example: To a conference on Middle East Peace the Americans (A) send five delegates, the Lebanese (L), Israelis (I), and Jordanians (J) each send three, and the Saudis (S) and Egyptians (E) two each, for a total of eighteen. Behind every delegate there shall stand a flag showing his nationality, like this:



Every morning the moderator, who believes in mixing it up, has the gofers arrange the 18 flags arbitrarily and then asks the delegates to sit in a chair in front of a flag of their nationality.

Problem: How many different arrangements of the eighteen flags are there?

Answer1: Let x denote this (as yet unknown) number. We use it to count the number of ways of seating the delegates ($18!$, of course). First, we choose an arrangement of flags (x); then we seat the 5 Americans in the five chairs with an American flag ($5!$ ways); then we seat the Lebanese, Israelis, and Jordanians in front of their flags ($3! \cdot 3! \cdot 3!$ ways); then we seat the Saudis and Egyptians ($2! \cdot 2!$ ways). By

the Rule of Product there are

$$x \cdot 5! \cdot 3! \cdot 3! \cdot 3! \cdot 2! \cdot 2!$$

ways of seating the delegates. This number equals $18!$**. Thus the number** x **of arrangements of flags is**

$$x = \frac{18!}{5! \cdot 3! \cdot 3! \cdot 3! \cdot 2! \cdot 2!}$$

This number is also written as

$$\binom{18}{5\ 3\ 3\ 3\ 2\ 2},$$

pronounced "eighteen choose 5,3,3,3,2,2."

Answer2: We first choose 5 **of the** 18 **seats for the** 5 **American flags. This can be done in**

$$\binom{18}{5} = \frac{18!}{5!13!} \quad \textbf{ways.}$$

Then we choose 3 of the remaining 13 seats for the 3 Lebanese flags. This can be done in

$$\binom{13}{3} = \frac{13!}{3!10!} \quad \textbf{ways.}$$

Then we choose 3 of the remaining 10 seats for the 3 Israeli flags. This can be done in

$$\binom{10}{3} = \frac{10!}{3!7!} \quad \textbf{ways.}$$

Keep going. Finally choose 2 of the remaining 2 seats for the 2 Egyptian flags. This can be done in

$$\binom{2}{2} = \frac{2!}{2!0!} \quad \textbf{ways.}$$

Therefore there are

$$\frac{18!}{5!13!} \cdot \frac{13!}{3!10!} \cdot \frac{10!}{3!7!} \cdots \frac{2!}{2!0!} \quad \textbf{ways.}$$

Cancel and arrive at

$$\frac{18!}{5! \cdot 3! \cdot 3! \cdot 3! \cdot 2! \cdot 2!} .$$

Theorem: Suppose you have n_1 indistinguishable widgets of type 1, n_2 indistinguishable widgets of type 2, $\cdots$, n_r indistinguishable widgets of type r, for a total of $n \stackrel{\text{def}}{=} n_1 + n_2 + \cdots + n_r$ widgets.
Then there are

$$\binom{n}{n_1 \; n_2 \; \cdots \; n_r} \stackrel{\text{def}}{=} \frac{n!}{n_1! \cdot n_2! \cdots n_r!}$$

distinguishable arrangements of these widgets.
Proof: Adapt either of the two preceding arguments.

Note: $\binom{n}{n_1\, n_2\, \cdots\, n_r}$ **is also the number of ways of splitting a set of size n into r subsets, of which the first has n_1 elements, the second has n_2 elements,$\cdots$, and the $r^{\mathbf{th}}$ has n_r elements, $n_1 + \cdots + n_r = n$.**

Note: $\binom{n}{k} = \binom{n}{k\ (n-k)}$

The Binomial Theorem

Let $a, b \in \mathbb{R}$, $n \in \mathbb{N}$. Then

$$(a+b)^n = \underbrace{(a+b)(a+b)(a+b)\cdots(a+b)}_{n \textbf{ factors}}$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

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Answer 1: **Apply the Binomial Formula with** $a = b = 1$

Answer 2: 2^n **is the number of subsets of a set** S **with** n **elements. These subsets fall into the classes of subsets with** 0 **elements** $[\binom{n}{0}]$**, subsets with** 1 **element** $[\binom{n}{1}]$**, subsets with** 2 **elements** $[\binom{n}{2}]$**,**$\cdots$**, subsets with** n **elements** $[\binom{n}{n}]$**.**

Example: **Why is** $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$**?**

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Example: Why is $\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0$?

Answer: Apply the Binomial Formula with $a = -1\ ,\ b = 1$.

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Answer: **Apply the Binomial Formula with** $a = -x$, $b = 1$ **and differentiate, then set** $x = 1$**:**

$$(-x+1)^n = \sum_{k=0}^{n}(-x)^k\binom{n}{k} \qquad \Longrightarrow$$

$$-n(-x+1)^{n-1} = -\sum_{k=0}^{n}(-x)^{k-1}k\binom{n}{k} \qquad \Longrightarrow$$

$$0 = \sum_{k=0}^{n}(-1)^{k-1}k\binom{n}{k} \ .$$

The Multinomial Theorem

Let $a_1, a_2, \cdots, a_r \in \mathbb{R}$**,** $n \in \mathbb{N}$**. Then**

$$(a_1 + a_2 + \cdots + a_r)^n$$

$$= \underbrace{(a_1 + a_2 + \cdots + a_r) \cdots (a_1 + a_2 + \cdots + a_r)}_{n \textbf{ factors}}$$

$$= \sum_{\substack{k_i \geq 0 \\ \sum_i k_i = n}} \binom{n}{k_1 \; k_2 \; \cdots \; k_r} a_1^{k_1} a_2^{k_2} \cdots a_r^{k_r} \, .$$

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Problem: How many summands are there in this sum?

Answer: As many as you can make. See the next section.
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Example: **How many arrangements do the letters TALLAHASSEE have?**
There are the groups [A], [E], [L], [H], [S], [T] with 3, 2, 2, 1, 2, 1, indistinguishable letters in them, respectively, for a total of 11. Thus there are

$$\binom{11}{3\ 2\ 2\ 1\ 2\ 1}$$

arrangements of these letters.

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Answer: **As many as you can make: first arrange the 8 letters EELLHSST, in one of**

$$\binom{8}{2\;2\;1\;2\;1}$$

ways. There are 9 places (which?) into which you can stick an A to complete the arrangement. So there are

$$\binom{8}{2\;2\;1\;2\;1} \times \binom{9}{3}$$

arrangements with no adjacent A's.

Selection with Repetition

Example: **Seven indistinguishable freshmen pile into a fast food joint that offers a very small menu: tacos (T), hamburgers (H), chicken sandwiches (C), and fish'nchips (F). How many different orders can they give the waitperson (WP)?**

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Answer: **As many as WP can record.**

T	H	C	F	Some Orders:
XXX	XX		XX	XXXIXXIIXX
	X X	XXXXX		IXXIXXXXXI
X	XX	X X	XX	XIXXIXXIXX
			XXX XXXX	IIIXXXXXXX

Answer:

$$\binom{7+(4-1)}{(4-1)} = \binom{7+(4-1)}{7}$$

Theorem: **There are**

$$\binom{n+\ k{-}1}{k{-}1} = \binom{n+\ k{-}1}{n}$$

ways of distributing n indistinguishable widgets over k distinguishable urns.

Example: How many solutions does the diophantine equation

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Answer: **None if** $n < 0$**; as many as there are ways of distributing** n **numbers** 1 **over the** k **variables** $x_1, \ldots, x_k$ **if** $n \geq 0$**:**

$$\#\{x_1+x_2+\cdots+x_k = n \ , \ x_i \geq 0\} = \binom{n + \ k-1}{k-1} \ .$$