Lecture 3

Review: **Principle 0, Principle 1, and**
Theorem: A set of size n has

$$\binom{n}{k} \stackrel{\text{def}}{=} \frac{n!}{k!(n-k)!}$$

subsets of size k. Arrangements, Binomial and Multinomial Theorems.

Selection with Repetition

Example: **Seven indistinguishable freshmen pile into a fast food joint that offers a very small menu: tacos (T), hamburgers (H), chicken sandwiches (C), and fish'nchips (F). How many different orders can they give the waitperson (WP)?**

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Answer: **As many as WP can record.**

T	H	C	F
XXX	XX		XX
	X X	XXXXX	
X	XX	X X	XX
			XXX XXXX

Some Orders:

XXXIXXIIXX

IXXIXXXXXI

XIXXIXXIXX

IIIXXXXXXX

Answer:

$$\binom{7+(4-1)}{(4-1)} = \binom{7+(4-1)}{7}$$

Theorem: **There are**

$$\binom{n+\ k{-}1}{k{-}1} = \binom{n+\ k{-}1}{n}$$

ways of distributing n indistinguishable widgets over k distinguishable urns.

Example: **How many solutions does the diophantine equation**

$x_1 + x_2 + \cdots + x_k = n\ ,\ \mathbb{Z} \ni x_i \geq 0\ ,$ **possess?**

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Example: **How many solutions does the diophantine equation**

$$x_1 + x_2 + \cdots + x_k = n\ ,\ \mathbb{Z} \ni x_i \geq 0\ ,$$ **possess?**

Answer: **None if $n < 0$; as many as there are ways of distributing n numbers 1 over**

the k variables $x_1, \ldots, x_k$ if $n \geq 0$:

$$\#\{x_1+x_2+\cdots+x_k = n\ ,\ x_i \geq 0\} = \binom{n+\ k-1}{k-1}\ .$$

Example: **How many solutions does the diophantine equation**

$$x_1 + x_2 + \cdots + x_k = n\ ,\ x_i \geq 3\ , \quad \textbf{possess?}$$

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Example: **How many solutions does the diophantine equation**

$$x_1 + x_2 + \cdots + x_k = n \ , \ x_i \geq 3 \ , \qquad \textbf{possess?}$$

Answer: **None if $n < 3k$. If $n \geq 3k$ subtract $3k$ from both sides:**

$$\#\{x_1 + x_2 + \cdots + x_k = n \ , \ x_i \geq 3\}$$

$$= \#\{y_1 + y_2 + \cdots + y_k = n - 3k \ , \ y_i \geq 0\}$$

$$= \binom{n - 3k \ + k-1}{k-1} = \binom{n - 2k - 1}{k-1} .$$

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$$x_1 + x_2 + \cdots + x_k < n \ , \ x_i \geq 0 \ , \quad \textbf{possess?}$$

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Answer:

$$\#\{x_1 + x_2 + \cdots + x_k < n \; , \; x_i \geq 0\}$$

$$= \#\{x_1 + x_2 + \cdots + x_k \leq n - 1 \; , \; x_i \geq 0\}$$

$$= \#\{x_1 + x_2 + \cdots + x_k + x_{k+1} = n - 1 \; , \; x_i \geq 0\}$$

$$= \binom{n - 1 \; + k}{k} \; .$$

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Answer:

$$\#\{x_1 + x_2 + \cdots + x_k < n \ , \ x_i \geq -2\}$$

$$= \#\{y_1 + y_2 + \cdots + y_k < n + 2k \ , \ y_i \geq 0\}$$

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$$= \#\{y_1 + y_2 + \cdots + y_k + y_{k+1} = n - 1 + 2k \ , \ y_i \geq 0\}$$

$$= \binom{n - 1 + 2k \ + k}{k} = \binom{n + 3k - 1}{k} \ .$$

Example: **Mother has four cookies, a heart–shaped one, a tree–shaped one, a plain disk, and a diamond–shaped one. She has 17 brown chocolate M&Ms. In how many ways can she place the M&Ms on the cookies in such a way that every Cookie gets at least two M&Ms?**

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Example: Mother has three doughnuts and six M&Ms: a pink one, a yellow one, a brown one, a red one, a green one, and a white one. In how many ways can she put the M&Ms on the doughnuts such that every doughnut gets at least one M&M?

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Answer: This is too hard for now. See later.