Elementary Logic

Logic is about Statements, also called Propositions. We shall use the following understanding of what a statement is:

Informal Definition: A statement or proposition is a pair consisting of a declaratory sentence together with a method to ascertain whether it is true or false.

Examples: In many statements, the method of verification is self-understood. For instance "The sun is shining" is understood to come with the verification scheme "go to the window and take a look–see." The

declaratory statement "$3 \times 3 = 9$" comes with a truth value "T" because generations of mathematicians who, as opposed to you, actually know what 3, $\times$, 9 actually are have checked it and found it to be true; you are inclined to accept their findings because you carried out checks with your fingers in kindergarten.

Is "What time is it?" a declaratory sentence?

declaratory statement "$3 \times 3 = 9$" comes with a truth value "T" because generations of mathematicians who, as opposed to you, actually know what 3, $\times$, 9 actually are have checked it and found it to be true; you are inclined to accept their findings because you carried out checks with your fingers in kindergarten.

Is "What time is it?" a declaratory sentence?

No, it is a question.

Is "Whoa!" a declaratory sentence?

declaratory statement "$3 \times 3 = 9$" comes with a truth value "T" because generations of mathematicians who, as opposed to you, actually know what 3, $\times$, 9 actually are have checked it and found it to be true; you are inclined to accept their findings because you carried out checks with your fingers in kindergarten.

Is "What time is it?" a declaratory sentence?

No, it is a question.

Is "Whoa!" a declaratory sentence?

No, it is an exclamation.

Is “The sun revolves about the earth” a declaratory sentence?

Is "The sun revolves about the earth" a declaratory sentence?

Yes; it has a subject and a verb, etc.
To define a verification scheme for it we must clarify what it means and how to verify that meaning. What does it mean? With which method of verification does it become a statement?

Is “The sun revolves about the earth” a declaratory sentence?

Yes; it has a subject and a verb, etc.
To define a verification scheme for it we must clarify what it means and how to verify that meaning. What does it mean? With which method of verification does it become a statement?

Is that statement then true?

Is “God exists” a statement? If so, what is its meaning and how does one verify it?

Is “God exists” a statement? If so, what is its meaning and how does one verify it?

After a bit of quibbling most theologians and ordinary people will agree on some definition of “God” that engenders a meaning of the sentence “God exists” common to them all. This still leaves the verification scheme. The greatest philosophers over the ages have come up with various arguments purporting to show that God exists. Many have come up with various arguments purporting to show that God does not exist. But this only shows that there is no universal verification scheme: if the argument of one of these great philosophers were entirely cogent, there would

have been no need for their followers to produce another one.

By our definition of what a statement is, "God exists" thus is not a statement, for the absence of a verification scheme.

Does all this mean that discussion of "God exists" is useless or superfluous?

Consider this: noone gets exercised over what we consider a fine statement like "$3 \times 3 = 9$." On the other hand many people have given their lives to defend or refute the claim "God exists" in its many variations. For many people nowadays the belief "God exists" or its negation "God does not exist" is fundamental to their understanding of themselves and their role in society, and even of how society should be structured and the world should be understood; it is of much bigger portent to them than "$3 \times 3 = 9$," or so they think.

Consider here again the statement "the sun is shining," whose verification scheme was supposed to be "look out the window and see." To a blind man or to a prisoner in a dungeon this prescription is useless.

At this point it might seem that our definition of a statement (one worth talking about) is too narrow.
Yet it is really the only way to analyze cogently the connection of various statements with each other. What does it mean to say "this statement implies that statement, and that implication is false?"

Here is where Axioms come in. The idea is this: if there are some declaratory sentences about whose meaning and verification is doubt in some people's mind let us declare the ones we are interested in as true or false, and see what logical conclusions we can draw from such axioms.
For instance, most all theologians and about 80% of Americans believe (but can't prove) that a benevolent God exists; so let us stipulate this as Axiom One of theology and argue from there. People who have had a relevation or upbringing as believers will accept the axiom as truth and with it all it logical consequences; people who disbelieve Axiom One will discard any logical consequences it has and accept all logical

consequences of its negation.

In the sequel we will shy away from any interpretations of axioms like this, but only concern ourselves with the mechanism of drawing conclusions from them, without any judgement of their merit.

Logical Connectives

First we make new statements from old; here are several ways:
1) Given a statement P, make its Negation $\neg P$ as follows: The declaration of $\neg P$ is the declaration of P with a "not" in front or inside; and the method of verification of $\neg P$ is this: check P by way of its given verification scheme: if P is true then $\neg P$ is false and if P is false then $\neg P$ is true.

Example:
"$3 \times 3 = 9$" **has negation** "$3 \times 3 \neq 9$".

Compound Statements:

2) Given two statements P and Q, make their Conjunction $P \wedge Q$ (read "P and Q") as follows:

a) the declaratory sentence of $P \wedge Q$ is the juxtaposition of the declaratory sentence of P with the declaratory sentence of Q with the word "and" between the two;

b) the method of verification of $P \wedge Q$ is this: check P and Q by their given methods; if both P and Q are true then $P \wedge Q$ is true, otherwise it is false.

Example: **Let MYC: "The Moon is made of Yellow Cheese" and PL: "The Pope is a Lady." Then MYC∧PL has declaration "The Moon is made of Yellow Cheese and The Pope is a Lady."**

3) Given two statements P and Q, make their Disjunction $P \vee Q$ (read "P or Q") as follows:

a) the declaratory sentence of $P \vee Q$ is the juxtaposition of the declaratory sentence of P with the declaratory sentence of Q with the word "or" between the two;

b) the method of verification of $P \vee Q$ is this: check P and Q by their given methods; if either or both of P and Q are true then $P \vee Q$ is true, otherwise (i.e. if both P and Q are false) it is false.

Example: **Is MYC$\vee$PL true?**

4) Given two statements P and Q, make the Implication $P \to Q$ (read "P implies Q", or "If P then Q") as follows:
a) the declaratory sentence of $P \to Q$ is the declaratory sentence of P followed by the word "implies" and the declaratory sentence of Q;
b) the method of verification of $P \to Q$ is this: check P and Q by their given methods; if P is false or Q is true then $P \to Q$ is true, otherwise (i.e. if P is true and Q is false) $P \to Q$ is false.
Example: **Is MYC→PL true?**

5) Given two statements P and Q, make the Equivalence $P \leftrightarrow Q$ (read "P iff Q") as follows:

a) the declaratory sentence of $P \leftrightarrow Q$ is the declaratory sentence of P followed by the word "iff" and the declaratory sentence of Q ("iff" is the lazy mathematician's way of writing "if and only if");

b) the method of verification of $P \leftrightarrow Q$ is this: check P and Q by their given methods; if P and Q have the same truth value then $P \leftrightarrow Q$ is true, otherwise it is false.

Example: Is MYC↔PL true?

Truth Tables

		not	and	or	implies	iff
P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Informal Definition:

Statements $P = P(p, q, r, s, \ldots)$ **that are made up from other statements** $p, q, r, s, \ldots$ **with the help of** $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ **etc. are called Compound Statements.**

All Other statements are Simple Statements.

A compound statement $P = P(p, q, r, s, \ldots)$ **that is true no matter what the truth values of its ingredients** $p, q, r, s, \ldots$ **are is a Tautology and we write** $P \iff T_0$**.**
If $P = P(p, q, r, s, \ldots)$ **is false no matter what the truth values of its ingredients** $p, q, r, s, \ldots$ **are then it is a Contradiction and we write** $P \iff F_0$**.**

Example:

$[p \vee \neg p] \iff T_0$ **and** $[p \wedge \neg p] \iff F_0$**.**

Example:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \to Q$	$\neg Q \to \neg P$	$Q \vee \neg P$
T	T	F	T	T	T	T	T
T	F	F	F	T	F	F	F
F	T	T	F	T	T	T	T
F	F	T	F	F	T	T	T

$$[P \to Q] \iff [\neg Q \to \neg P] \iff [Q \vee \neg P]$$

Therefore

$$(p \longrightarrow q) \leftrightarrow (\neg q \longrightarrow \neg p) \ ,$$

$$(p \longrightarrow q) \leftrightarrow (q \vee \neg p) \ , \qquad \textbf{and}$$

$$(\neg q \longrightarrow \neg p) \leftrightarrow (q \vee \neg p)$$

are tautologies.

To say that an equivalence $P(p, q, \ldots) \leftrightarrow Q(p, q, \ldots)$ is a tautology is the same as saying that $P(p, q, \ldots)$ and $Q(p, q, \ldots)$ have the same truth values for any constellation of truth values of their ingredients $p, q, \ldots$; in this case we say that $P(p, q, \ldots)$ and $Q(p, q, \ldots)$ are Logically Equivalent and write

$$P(p, q, \ldots) \iff Q(p, q, \ldots) \,.$$

Examples: 1) $p \iff (\neg\neg p)$.

Let P, Q be statements.

To say $P = Q$ means that P and Q have the same declaratory sentences and the same verification scheme.

To say $P \iff Q$ means that the statement $P \leftrightarrow Q$ is a tautology.

The Laws of Logic

$\neg\neg P$	$\iff$	P	**Law of double negation**
$\neg(P \vee Q)$	$\iff$	$\neg P \wedge \neg Q$	**de Morgan's law**
$\neg(P \wedge Q)$	$\iff$	$\neg P \vee \neg Q$	**de Morgan's law**
$P \vee Q$	$\iff$	$Q \vee P$	**Commutative Law**
$P \wedge Q$	$\iff$	$Q \wedge P$	**Commutative Law**
$P \vee (Q \vee R)$	$\iff$	$(P \vee Q) \vee R$	**Associativity**
$P \wedge (Q \wedge R)$	$\iff$	$(P \wedge Q) \wedge R$	**Associativity**
$P \wedge P$	$\iff$	P	**Idempotent Law**
$P \vee P$	$\iff$	P	**Idempotent Law**
$P \vee F_0$	$\iff$	P	**Identity Law**
$P \wedge T_0$	$\iff$	P	**Identity Law**

$$P \wedge \neg P \iff F_0$$ **Inverse Law**

$$P \vee \neg P \iff T_0$$ **Inverse Law**

$$P \vee T_0 \iff T_0$$ **Domination**

$$P \wedge F_0 \iff F_0$$ **Domination**

$$P \vee (P \wedge Q) \iff P$$ **Absorption**

$$P \wedge (P \vee Q) \iff P$$ **Absorption**

$$P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$$ **Distributive Law**

$$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$$ **Distributive Law**

Example: Using Logical Equivalence to Analyze a Switching Network click here.

More About Implications

P	Q	Implication $P \to Q$	Contrapositive $\neg Q \to \neg P$	Converse $Q \to P$	Inverse $\neg P \to \neg Q$	negation $\neg(P \to Q)$	negation $P \wedge \neg Q$
T	T	T	T	T	T	F	F
T	F	F	F	T	T	T	T
F	T	T	T	F	F	F	F
F	F	T	T	T	T	F	F

The negation of $P \to Q$ is NOT an implication!!!!!!!

The negation of "P implies Q" is "P yet not Q" or "P but not Q." ["Yet" and "but" are but emphatic versions of "and."]

Examples: **Negate "If the sun shines I will ride my bicycle."**

More About Implications

P	Q	Implication $P\to Q$	Contrapositive $\neg Q\to\neg P$	Converse $Q\to P$	Inverse $\neg P\to\neg Q$	negation $\neg(P\to Q)$	negation $P\wedge\neg Q$
T	T	T	T	T	T	F	F
T	F	F	F	T	T	T	T
F	T	T	T	F	F	F	F
F	F	T	T	T	T	F	F

The negation of $P \to Q$ is NOT an implication!!!!!!!

The negation of "P implies Q" is "P yet not Q" or "P but not Q." ["Yet" and "but" are but emphatic versions of "and."]

Examples: **Negate "If the sun shines I will ride my bicycle."**

"The sun shines but I won't ride my bike."

Negate P:“If the moon is made of yellow cheese then the pope is a lady.”

Negate P:"If the moon is made of yellow cheese then the pope is a lady."

$\neg P \iff$ **"The moon is made of yellow cheese yet the pope is not a lady."**

Negate P:"If the moon is made of yellow cheese then the pope is a lady."

$\neg P \iff$ **"The moon is made of yellow cheese yet the pope is not a lady."**

Which of the preceding two statements is true?

Substitution Rules

1) Suppose $P = P(p, q, \ldots)$ is a tautology. Replace every occurrence of the primitive statement p by the same statement p', thus creating the new statement $P' = P'(p', q, \ldots)$. Then P' is also a tautology.

2) Suppose $P = P(p, q, \ldots)$ is a compound statement. Suppose you replace the ingredient statement p at several occurrences (not necessarily all) by some statement $p' \iff p$, thus creating the new statement P'. Then $P \iff P'$.

Example Negate and simplify $(p \vee q) \to r$.

$$\neg[(p \vee q) \to r] \iff \neg(p \vee q) \vee r$$

, thus

$$\neg[(p \vee q) \to r] \iff \neg[\neg(p \vee q) \vee r] \iff$$

$$\neg\neg(p \vee q) \wedge \neg r \iff$$

$$(p \vee q) \wedge \neg r \ .$$

In summary:

$$\neg[(p \vee q) \to r] \iff (p \vee q) \wedge \neg r \ .$$

Logical Implications

Consider compound statements $P_1, \ldots P_n, Q$, all made of the same ingredients. An implication of the form

$$(P_1 \wedge \cdots \wedge P_n) \rightarrow Q \qquad (A)$$

is called an argument. $P_1, \ldots P_n$ are the Premises of the argument, Q its Conclusion. If (A) is a tautology then the argument (A) is Valid, and the implication (A) is called a Logical Implication; this is written

$$(P_1 \wedge \cdots \wedge P_n) \implies Q \qquad (I)$$

and pronounced "$P_1, \ldots P_n$ logically imply Q."

Notabene:

$$(P_1 \wedge \cdots \wedge P_n) \to Q \qquad (A)$$

is just a statement concocted from $P_1, \ldots P_n, Q$, in the form of an implication.

$$(P_1 \wedge \cdots \wedge P_n) \implies Q \qquad (I)$$

is an assertion about (A), namely that (A) is always true, that it is a tautology.

Example: From the simple statements

p	:	A will pass the course
r	:	A plays racketball
s	:	A studies

make the compound statements

P	:	$s \to p$	:	if A studies she passes
Q	:	$\neg r \to s$	:	if A doen't play she studies
R	:	$\neg p$	:	A fails the course

and the argument

$$(P \wedge Q \wedge R) \to r \textbf{ or}$$
$$\{(s \to p) \wedge (\neg r \to s) \wedge \neg p\} \to r \ .$$

Show that this is, in fact, a logical implication.

Example: From the simple statements

$$\begin{aligned} p &: \quad \text{A will pass the course} \\ r &: \quad \text{A plays racketball} \\ s &: \quad \text{A studies} \end{aligned}$$

make the compound statements

$$\begin{array}{llll} P : & s \rightarrow p & : & \text{if A studies she passes} \\ Q : & \neg r \rightarrow s & : & \text{if A doen't play she studies} \\ R : & \neg p & : & \text{A fails the course} \end{array}$$

and the argument

$$(P \wedge Q \wedge R) \rightarrow r \text{ or}$$
$$\{(s \rightarrow p) \wedge (\neg r \rightarrow s) \wedge \neg p\} \rightarrow r \ .$$

Show that this is, in fact, a logical implication.
To see this either make a truth table for it or show that there is no constellation of

truth values of the p, r, s for which the implication is false. Namely, in such a constellation we would have:
a) r **false**
b) $\{(s \to p) \wedge (\neg r \to s) \wedge \neg p\}$ **true;**
and therefore $s \to p$**,** $\neg r \to s$**, and** $\neg p$ **all true.**
c) p **false;** s **false;** $\neg r$ **false, hence** r **true.**
Because of a), there is no such constellation.