

# Elementary Logic

**Review Propositions, Logical Connectives: Conjunction, Disjunction, Implication, Equivalence. Truth Tables. Tautologies, Logical Equivalence, and Contradictions. The Laws of Logic. Substitution Rules.**

# More About Implications

$P$	$Q$	Implication $P \rightarrow Q$	Contra- positive $\neg Q \rightarrow \neg P$	Con- verse $Q \rightarrow P$	In- verse $\neg P \rightarrow \neg Q$	ne- gation $\neg(P \rightarrow Q)$	ne- gation $P \wedge \neg Q$
$T$	$T$	$T$	$T$	$T$	$T$	$F$	$F$
$T$	$F$	$F$	$F$	$T$	$T$	$T$	$T$
$F$	$T$	$T$	$T$	$F$	$F$	$F$	$F$
$F$	$F$	$T$	$T$	$T$	$T$	$F$	$F$

The negation of  $P \rightarrow Q$  is NOT an implication!!!!!!

The negation of “P implies Q” is  
“P yet not Q” or “P but not Q.”

[“Yet” and “but” are but emphatic ver-  
sions of “and.”]

**Examples:** Negate “If the sun shines I will  
ride my bicycle.”

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sions of “and.”]

**Examples:** Negate “If the sun shines I will  
ride my bicycle.”

“The sun shines but I won’t ride my bike.”

Negate P: “If the moon is made of yellow cheese then the pope is a lady.”

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$\neg P \iff$  “The moon is made of yellow cheese yet the pope is not a lady.”

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Which of the preceding two statements is true?

# Substitution Rules

1) Suppose  $P = P(p, q, \dots)$  is a tautology. Replace **every** occurrence of the **primitive** statement  $p$  by the **same** statement  $p'$ , thus creating the new statement  $P' = P'(p', q, \dots)$ . Then  $P'$  is also a tautology.

2) Suppose  $P = P(p, q, \dots)$  is a compound statement. Suppose you replace the ingredient statement  $p$  at several occurrences (not necessarily all) by some statement  $p' \iff p$ , thus creating the new statement  $P'$ . Then  $P \iff P'$ .

**Example** Negate and simplify  $(p \vee q) \rightarrow r$ .

$$\begin{aligned}(p \vee q) \rightarrow r &\iff \neg(p \vee q) \vee r \quad , \text{ thus} \\ \neg[(p \vee q) \rightarrow r] &\iff \neg[\neg(p \vee q) \vee r] \iff \\ \neg\neg(p \vee q) \wedge \neg r &\iff \\ (p \vee q) \wedge \neg r . &\quad \text{In summary:} \\ \neg[(p \vee q) \rightarrow r] &\iff (p \vee q) \wedge \neg r .\end{aligned}$$

# Logical Implications

Consider compound statements  $P_1, \dots, P_n, Q$ , all made of the same ingredients. An implication of the form

$$(P_1 \wedge \dots \wedge P_n) \rightarrow Q \quad (A)$$

is called an **argument**.  $P_1, \dots, P_n$  are the **Premises** of the argument,  $Q$  its **Conclusion**. If  $(A)$  is a tautology then the argument  $(A)$  is **Valid**, and the implication  $(A)$  is called a **Logical Implication**; this is written

$$(P_1 \wedge \dots \wedge P_n) \implies Q \quad (I)$$

and pronounced “ $P_1, \dots, P_n$  logically imply  $Q$ .”

**Notabene:**

$$(P_1 \wedge \cdots \wedge P_n) \rightarrow Q \quad (A)$$

is just a statement concocted from  $P_1, \dots, P_n, Q$ , in the form of an implication.

$$(P_1 \wedge \cdots \wedge P_n) \implies Q \quad (I)$$

is an assertion about  $(A)$ , namely that  $(A)$  is always true, that it is a tautology.

**Example:** From the simple statements

$p$  :        A will pass the course

$r$  :        A plays racketball

$s$  :        A studies

make the compound statements

$P$  :  $s \rightarrow p$     :    if A studies she passes

$Q$  :  $\neg r \rightarrow s$  :    if A doesn't play she studies

$R$  :  $\neg p$          :    A fails the course

and the argument

$$(P \wedge Q \wedge R) \rightarrow r \text{ or}$$
$$\{(s \rightarrow p) \wedge (\neg r \rightarrow s) \wedge \neg p\} \rightarrow r .$$

Show that this is, in fact, a logical implication.

**Example:** From the simple statements

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Show that this is, in fact, a logical implication.

To see this either make a truth table for it or show that there is no constellation of

truth values of the  $p, r, s$  for which the implication is false. Namely, in such a constellation we would have:

a)  $r$  false

b)  $\{(s \rightarrow p) \wedge (\neg r \rightarrow s) \wedge \neg p\}$  true;

and therefore  $s \rightarrow p$ ,  $\neg r \rightarrow s$ , and  $\neg p$  all true.

c)  $p$  false;  $s$  false;  $\neg r$  false, hence  $r$  true.

Because of a), there is no such constellation.

# Rules of Inference

Here are a few Logical Implications that because of their ubiquitous nature have the name **Rules of Inference**.

1) **Modus Ponens**  $P \wedge (P \rightarrow Q) \implies Q$ .

2) **Rule of Syllogism**

$(P \rightarrow Q) \wedge (Q \rightarrow R) \implies (P \rightarrow R)$ .

3) **Modus Tollens**  $\{(P \rightarrow Q) \wedge \neg Q\} \implies \neg P$ .

5) **Rule of Disjunctive Syllogism**

$\{(P \vee Q) \wedge \neg P\} \implies Q$ .

6) **Rule of Contradiction**  $\{\neg P \rightarrow F_0\} \implies P$ .

See the book for more or look [here](#).

# Quantifiers

Consider the declarative sentence

$$“x > 0” \quad (O)$$

So far this is not a statement, as no verification scheme has been supplied. Specifically, we have not been told what the value of the variable  $x$  is, so while we generally know how to verify “ $>$ ”, we cannot use that knowledge. Sentences like the above, that contain unspecified variables, are known as **Open Statements**.

We can make a statement from  $(O)$  by use of the **quantifiers**

$\forall$  “for all” and  
 $\exists$  “for some” or “there exists”,

turning  $(O)$  into

$\forall x \quad x > 0$  or  
 $\exists x \quad x > 0$ , respectively.  $(*)$

This is still not satisfactory. Imagine we had been talking about cows for a while and then someone issues the sentence  $(*)$ . Trying to check whether a cow “ $> 0$ ” is like applying a lawn mower to your mother - don't. We need to specify from which set the variable  $x$  is to be taken.

This can be done in two ways:

1) We declare a **Universe**  $\mathcal{U}$  from which all variables appearing are to be taken. For instance, if we declare  $\mathcal{U} \stackrel{\text{def}}{=} \mathbb{R}$ , then  $(*)$  means “There exists a real number  $x$  with  $x > 0$ .”

2) We specify within the quantified sentence where the variable comes from:

$$\exists x \in \mathbb{R} \ x > 0 \text{ or } \exists x \in \mathbb{R} \ni x > 0$$

[The lowered  $\ni$  is read “such that.”]

For clarity’s sake we will usually go route 2).

**Universal Quantifier:** Given an open statement  $P(x)$  and a set  $\mathcal{U}$  we make a statement

$$\forall x \in \mathcal{U} \ P(x) \quad (*_{\forall})$$

involving the **Universal Quantifier**  $\forall$  as follows: The declaratory sentence of  $(*_{\forall})$  is the

juxtaposition of “For all  $x$  in  $\mathcal{U}$ ” with  $P(x)$ . The verification scheme [**Rules of Universal Specification & Generalization** of the book] of  $(*_{\forall})$  is this:  $(*_{\forall})$  is true if for every (any arbitrary) element  $x \in \mathcal{U}$ ,  $P(x)$  is true.

**Existential Quantifier:** Given an open statement  $P(x)$  and a set  $\mathcal{U}$  we make a statement

$$\exists x \in \mathcal{U} \ni P(x) \quad (*\exists)$$

involving the **Existential Quantifier**  $\exists$  as follows: The declaratory sentence of  $(*\exists)$  is the juxtaposition of “There exists an  $x$  in  $\mathcal{U}$  such that” with  $P(x)$ . The verification scheme for  $(*\exists)$  is the obvious one:  $(*\exists)$  is true if  $\mathcal{U}$  contains an element  $e$  such that  $P(e)$  is true.

Clearly, the negation of  $(*\exists)$  is

$$\neg [\exists x \in \mathcal{U} \ni P(x)] \iff [\forall x \in \mathcal{U} \neg P(x)]$$

and

$$\neg [\forall x \in \mathcal{U} P(x)] \iff [\exists x \in \mathcal{U} \ni \neg P(x)] .$$

**Example:** Find the negation of “All lawyers are greedy.”

**Example:** Find the negation of “All lawyers are greedy.”

“All lawyers are not greedy?”

**Example:** Find the negation of “All lawyers are greedy.”

“All lawyers are not greedy?”

No. “Some lawyer is not greedy.”

**Example:** Find the negation of “All is lost.”

Generalize to open statements  $P(x, y, z, \dots)$  with several undeclared variables  $x, y, z, \dots$

**Example:** Do

$$\forall x \in \mathbb{R} \quad \exists y \in \mathbb{R} \ni x + y = 6 \quad \text{and}$$

$$\exists y \in \mathbb{R} \quad \forall x \in \mathbb{R} \ni x + y = 6 \quad \text{and}$$

mean the same thing?

**Example:** Given that  $f$  is a fixed function, compute the negation of

$$\forall x \in \mathbb{R} \forall \epsilon > 0 \exists \delta > 0 \ni \forall x' \in \mathbb{R} \\ |x - x'| < \delta \rightarrow |f(x) - f(x')| < \epsilon$$

$$\begin{aligned}
& \neg \left[ \forall x \in \mathbb{R} \forall \epsilon > 0 \exists \delta > 0 \ni \forall x' \in \mathbb{R} \right. \\
& \left. |x - x'| < \delta \rightarrow |f(x) - f(x')| < \epsilon \right] \iff \\
& \neg \left[ \forall x \in \mathbb{R} \left( \forall \epsilon > 0 \exists \delta > 0 \ni \forall x' \in \mathbb{R} \right. \right. \\
& \left. \left. [|x - x'| < \delta \rightarrow |f(x) - f(x')| < \epsilon] \right) \right] \iff \\
& \exists x \in \mathbb{R} \neg \left( \forall \epsilon > 0 \left\{ \exists \delta > 0 \ni \forall x' \in \mathbb{R} \right. \right. \\
& \left. \left. [|x - x'| < \delta \rightarrow |f(x) - f(x')| < \epsilon] \right\} \right) \iff \\
& \exists x \in \mathbb{R} \exists \epsilon > 0 \neg \left\{ \exists \delta > 0 \ni \forall x' \in \mathbb{R} \right. \\
& \left. [|x - x'| < \delta \rightarrow |f(x) - f(x')| < \epsilon] \right\} \iff \\
& \exists x \in \mathbb{R} \exists \epsilon > 0 \ni \forall \delta > 0 \neg \left\{ \forall x' \in \mathbb{R} \right. \\
& \left. [|x - x'| < \delta \rightarrow |f(x) - f(x')| < \epsilon] \right\} \iff
\end{aligned}$$

$$\begin{aligned} & \exists x \in \mathbb{R} \exists \epsilon > 0 \ni \forall \delta > 0 \exists x' \in \mathbb{R} \ni \\ & \neg[|x - x'| < \delta \rightarrow |f(x) - f(x')| < \epsilon] \quad \iff \\ & \exists x \in \mathbb{R} \exists \epsilon > 0 \ni \forall \delta > 0 \exists x' \in \mathbb{R} \ni \\ & [|x - x'| < \delta \wedge |f(x) - f(x')| \geq \epsilon] . \end{aligned}$$

**Notabene:**

**At the end of the statement**

$\forall x \in \mathcal{U} P(x)$  or  $\exists x \in \mathcal{U} \ni P(x)$

**the name  $x$  has expired.**

**Example:** ... we have proved that

$$\forall x \in \mathbb{R} \quad x^2 \geq 0 .$$

Therefore we can take the square root  $\sqrt{x^2}$ .

The previous statement is open and cannot be checked.