Set Theory

Set theory is about the membership relation $\in$; $x \in S$ means x is a member of the set S. This is another informal definition. The statement "S is a set" consists of this sentence plus a verification scheme that permits us to decide whether any widget is or is not a member of S. [That verification scheme depends, of course, on S.]

Example: **A catalogue is a printed list of books. Consider the collection of all catalogues that do not list themselves, and the catalogue that lists that collection.**

Does that catalogue list itself?

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No. For if it listed itself it would list a catalogue that lists itself, contradicting its own definition.
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Does that catalogue list itself?
Yes. For if it didn't list itself it would be a catalogue not listing itself so it belongs into itself.

Example: **An big army comes with many barbers. Among the barbers are those who don't shave themselves. Consider the barber who shaves all those barbers who don't shave themselves. Does he shave himself?**

Example: Consider the set of all sets who are not members of themselves. Is this set a member of itself?

Example: Consider the set of all sets who are not members of themselves. Is this set a member of itself?

This "set" is no set. There is no "set of all sets."

These examples show that we have to be careful with the membership relation $\in$. The way mathematics goes around this quandary is to stipulate that there is one set $\emptyset$ (the Empty Set that has no elements), and to provide a number of operations that produce new sets from old. For instance, $\{\emptyset\}$ is a new set, whose single element is the empty set. $\{\{\{\emptyset\}\}, \{\emptyset\}\}$ is a set with two elements, etc. Among the permissible operations are forming subsets, unions and intersections, cartesian prod-

ucts, and power sets, all to be explained below in suitable order.

The collection of sets one can form using these operations is huge, but it does not form a set. None of these operations lead to a set that is an element of itself. Among the sets obtained by permissible operations are the

Natural Numbers $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$
Strictly Positive Natural Numbers

$$\mathbb{N}_* = \{1, 2, 3, \ldots\}$$

Integers $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$
Rational Numbers

$$\mathbb{Q} = \{m/n : m, n \in \mathbb{Z}, n \neq 0\}$$

Real Numbers $\mathbb{R}$, **Complex** Numbers $\mathbb{C}$

Positive Real Numbers

$$\mathbb{R}_+ \stackrel{\text{def}}{=} \{r \in \mathbb{R} : r \geq 0\}$$

Non–zero Real Numbers

$$\mathbb{R}_* \stackrel{\text{def}}{=} \{r \in \mathbb{R} : r \neq 0\}$$

Strictly Positive Real Numbers

$$\mathbb{R}_{+*} \stackrel{\text{def}}{=} \{r \in \mathbb{R} : r > 0\}$$

and so on.
Note the various ways we describe a set above. Very frequently used is this description:

$$\{a \in A : p(a)\} \quad :$$

Given a set A **and an open statement** $p(x)$ **it describes the set of all** $a \in A$ **such that** $p(a)$ **is true.**

Usually one agrees on any one occasion to talk only about one specific set large enough to satisfy all needs, and that set is (for the purpose of that conversation) then called the Universal Set $\mathcal{U}$.

Definition: **Two sets A, B are equal:**

$$A = B \textbf{ if } \quad \forall\, x \in \mathcal{U}\; x \in A \iff x \in B\,.$$

A set A is a Subset of a set B:

$$A \subseteq B \textbf{ if } \forall x \;\; x \in A \implies x \in B\,.$$

Here and later "$\forall x$" means "$\forall x \in \mathcal{U}$." Frequently in the literature $A \subset B$ also means $A \subseteq B$; the book takes it to mean "A is a subset of B but not equal to B. If we want to express this we shall write

$$A \subsetneqq B \iff A \subseteq B \;\wedge\; A \neq B\,.$$

Example: For $A, B, C \subseteq \mathcal{U}$

$$A = B \iff [A \subseteq B \wedge B \subseteq A] .$$

$$A \subseteq B \ \wedge \ B \subseteq C \implies A \subseteq C .$$

$$\emptyset \subseteq A ,$$

because $\forall x \in \mathcal{U} \ \ x \in \emptyset \implies x \in A$.

There is only one empty set. For if $\emptyset, \emptyset'$ are two empty sets, they have exactly the same elements:

$$\forall x \in \mathcal{U} \ \ x \in \emptyset \iff x \in \emptyset' .$$

Call $\emptyset$ the **void set** if you will, but **don't call it a null set**. "Null set" is a notion from measure theory; using this phrase also in set theory is burocratese language creep.

Definition: The **Power Set** $\mathcal{P}[S]$ of a set S is the set of all of its subsets.

Example: **If the set S has size $|S| = n$ then its power set has size $|\mathcal{P}[S]| = 2^n$.**

Example: Pascal's triangle **Because a set $S = \{s_0, s_1, \ldots, s_n\}$ has $\binom{n+1}{k}$ subsets of size k, $\binom{n}{k-1}$ of which contain s_0 and $\binom{n}{k}$ of which don't,**

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} .$$

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$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Set Operations

Here are several ways of making new sets from old. Let $A, B \subset \mathcal{U}$.

$$\overline{A} \stackrel{\text{def}}{=} \{x \in \mathcal{U} : x \notin A\}$$

is the Complement of A

$$B \setminus A \stackrel{\text{def}}{=} = B - A \stackrel{\text{def}}{=} \{x \in B : x \notin A\}$$

is the Set Difference B minus A

$$A \cup B \stackrel{\text{def}}{=} \{x \in \mathcal{U} : x{\in}A \vee x{\in}B\}$$

is the Union of A and B

$$A \cap B \stackrel{\text{def}}{=} \{x \in \mathcal{U} : x{\in}A \wedge x{\in}B\}$$

is the Intersection of A and B,

$$A \Delta B \stackrel{\text{def}}{=} \{x \in \mathcal{U} : x{\in}A \wedge x{\notin}B \ \vee \ x{\in}B \wedge x{\notin}A\}$$

is their Symmetric Difference

$$A \times B \stackrel{\text{def}}{=} \{(a, b) : a \in A \ , \ b \in B\}$$

is the cartesian Product of A & B

$$\mathcal{P}[A] \stackrel{\text{def}}{=} \{C \subseteq \mathcal{U} : C \subseteq A\}$$

is the Power Set of A.

Definition: **Let $\mathcal{F}$ be a family of sets. Its Union is the set**

$$\bigcup \mathcal{F} \stackrel{\text{def}}{=} \{x \in \mathcal{U} : \exists\, F \in \mathcal{F} \ni x \in F\}.$$

Its Intersection is the set

$$\bigcap \mathcal{F} \stackrel{\text{def}}{=} \{x \in \mathcal{U} : \forall\, F \in \mathcal{F} \;\; x \in F\}.$$

Once in a while the family $\mathcal{F}$ comes Indexed: There is a set I, the Index Set, such that $\mathcal{F} = \{F_i : i \in I\}$. In that case one writes also

$$\bigcap \mathcal{F} = \bigcap_{i \in I} F_i \text{ and } \bigcup \mathcal{F} = \bigcup_{i \in I} F_i \ ,$$

respectively.

Definition: $A, B \subset \mathcal{U}$ **are <mutually> disjoint if $A \cap B = \emptyset$.**

Example: $A \setminus B = A \cap \overline{B}$.

Example: $A, B \subset \mathcal{U}$ **are disjoint iff**

$$A \cup B = A \Delta B \ .$$

Proof of $\Longrightarrow$:

$$\begin{aligned} \forall x \in \mathcal{U} \quad x \in A \cup B &\Longleftrightarrow x \in A \vee x \in B \\ &\Longleftrightarrow x \in A \setminus B \vee x \in B \setminus A \qquad \textbf{why?} \\ &\Longleftrightarrow x \in A \Delta B \ . \end{aligned}$$

Proof of $\Longleftarrow$: Assume that $A \cup B = A \Delta B$ and try to prove that $A \cap B = \emptyset$. By Way of Contradiction (BWoC) assume $A \cap B \neq \emptyset$ and let $x \in A \cap B$. Then $x \in A \cup B$ but $x \notin A \Delta B$. **QED**

Example: Let $A, B \subseteq \mathcal{U}$. Then

$$A\times B=B\times A \iff [A=\emptyset \vee B=\emptyset \vee A=B] .$$

Proof of $\Longleftarrow$: Obvious.

Proof of $\implies$:

$$A\times B=B\times A \to [A=\emptyset \vee B=\emptyset \vee A=B]$$

is logically equivalent to

$$\left[A\times B=B\times A \wedge A\neq\emptyset \wedge B\neq\emptyset\right] \to A=B ,$$

so we prove that this is a tautology.

To prove: Under assumptions $[\cdots]$, $A \subseteq B$ and $B \subseteq A$.

To prove: $\forall\, a \in \mathcal{U}\; a\in A \implies a\in B$ etc.

Let $a \in A$. Since $B \neq \emptyset$, $\exists b \in B$. Then $(a, b) \in A \times B$. Since $A \times B = B \times A$, $(a, b) \in B \times A$. Thus $a \in B$, and $A \subseteq B$. $B \subseteq A$ is proved similarly. Hence $A = B$. QED