Induction

The Universe shall be the set $\mathbb{Z}$ **of integers. It contains the natural numbers** $\mathbb{N} = \{0, 1, 2, \ldots\}$**. We assume that we know all about addition and multiplication of integers, and about their order. For instance, for all** $a, b, c \in \mathbb{Z}$

$$a+(b+c) = (a+b)+c \qquad a(bc) = (ab)c$$

$$a+b = b+a \qquad ab = ba$$

$$a(b+c) = ab+ac$$

$$\exists 0 \ni \forall x \;\; x+0 = x \qquad \exists 1 \ni \forall x \;\; x1 = x$$

$$\forall x \; \exists -x \ni x+-x = 0$$

$$a \le b \Longrightarrow a+c = b+c \quad (a \le b \wedge p \ge 0) \Longrightarrow ap \le bp$$

Well Ordering Principle (WOP):

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How does this differ from a well ordering principal?

It occurs that one wishes to prove a whole infinite slew of statements

$$S_0, S_1, S_2, \cdots .$$

For instance, for every natural number $n \in \mathbb{N}$ **let**

$$S_n\colon \sum_{k=1}^{n} k = \frac{n(n+1)}{2} .$$

To paraphrase: the sum of the first n **natural numbers is** $n(n+1)/2$**. If you try to prove these statements one by one, you'll never have a beer again, and still you won't get through.**

That is where <mathematical> Induction comes in:

Theorem: **Let** $S_0, S_1, \cdots$ **be a sequence of statements.**
If
a) the first statement S_0 **is true and**
b) for all $n = 0, 1, 2, 3 \ldots\ S_n \implies S_{n+1}$**,**
then all the statements $S_0, S_1, \cdots$ **are true.**

Proof: **BWoC assume the statements** S_n **are not all true. Then the set**

$$I \stackrel{\text{def}}{=} \{n \in \mathbb{N} : S_n \textbf{ is false}\}$$

is not empty and has a least element, say l**. Now** $l > 0$**, because of the basic step a). Therefore** $l - 1 \in \mathbb{N}$ **and** S_{l-1} **is true. Why?**

By the Induction Step b), $S_l = S_{(l-1)+1}$ is true, a plain contradiction QED

To paraphrase the proof: if some of the S_n were false, then look at the first false statement S_l; it is not the first statement S_0, so the statement before S_l is one of the S_n and is true, and it logically implies S_l, because of b). This cannot be.

Example: **Consider the statements**

$$S_n: \sum_{k=1}^{n} k = \frac{n(n+1)}{2}\,. \qquad n = 1, 2, 3, \cdots$$

The first one, S_1, says that

$$\sum_{k=1}^{1} k = \frac{1(1+1)}{2}\,,$$

which is true, since both sides of the equation equal 1. Next the induction step: to prove that $S_n \implies S_{n+1}$ for $n = 1, 2, 3, \ldots$.

$$\sum_{k=1}^{n+1} k = \underbrace{\sum_{k=1}^{n} k}_{} + (n+1)$$

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{(n+2)(n+1)}{2} = \frac{(n+1)\big((n+1)+1\big)}{2}.$$

By Induction, all the S_n are true. QED

Gauß in Kindergarten:

$$\begin{array}{lllll} s & = 1 & +2 & + \cdots & +100 \\ s & = 100 & +99 & + \cdots & +1 \\ \hline 2s & = 100 \times 101 & & & \end{array}$$

Example: Prove that

$$\sum_{k=1}^{n} k = \frac{n^2+n+2}{2} \,. \qquad\qquad n = 1, 2, \ldots$$

Proof of $S_n \implies S_{n+1}$:

$$\begin{aligned} \textstyle\sum_{k=1}^{n+1} k &= \textstyle\sum_{k=1}^{n} k + (n+1) \\ &= \tfrac{n^2+n+2}{2} + \tfrac{2(n+1)}{2} \\ &= \tfrac{n^2+n+2+2n+2}{2} = \tfrac{(n+1)^2+(n+1)+2}{2} \,. \end{aligned}$$

So what's wrong?

Example: Prove that

$$\sum_{k=1}^{n} k = \frac{n^2 + n + 2}{2} \,. \qquad\qquad n = 1, 2, \ldots$$

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So what's wrong?

Example: Prove the statements

$$S_n : \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \,.$$

$$n = 1, 2, 3, \cdots$$

The first statement is

$$S_1 : \sum_{k=1}^{1} k^2 = \frac{1(1+1)(2+1)}{6} \, ,$$

true since both sides evaluate to 1**.**

Next the induction step: to prove that $S_n \implies S_{n+1}$ **for** $n = 1, 2, 3, \cdots$**.**

$$\begin{aligned} \textstyle\sum_{k=1}^{n+1} k^2 &= \textstyle\sum_{k=1}^{n} k^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} \\ &= \frac{(n+1)(2n^2+n+6(n+1))}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \\ &= \frac{\big(n+1\big)\big((n+1)+1\big)\big(2(n+1)+1\big)}{6} \, , \end{aligned}$$

which proves the induction step. QED

Theorem (Induction, Alternate Form): Let $S_{n_0}, S_{n_0+1}, S_{n_0+2}, \cdots$ be statements, indexed by the integers $n_0, n_0+1, n_0+2, \cdots$. If
a) the statements in an initial segment

$$\mathcal{S} = \{S_{n_0}, S_{n_0+1}, \cdots, S_{n_1}\}$$

are all true and
b) whenever an initial segment

$$\{S_{n_0}, S_{n_0+1}, \cdots, S_{n_2}\}$$

containing $\mathcal{S}$ contains only true statements then the next statement S_{n_2+1} is also true
Then the statements $S_{n_0}, S_{n_0+1}, S_{n_0+2}, \cdots$ are all true.

Example: For $n = 14, 15, \cdots$ consider the statements

S_n: n is a sum of 3's and 8's.

The initial segment $\{S_{14}, S_{15}, S_{16}\}$ **is easily verified to consist only of true statements. Let us then prove**

$$[S_{14} \wedge \ldots \wedge S_n] \implies S_{n+1}$$

for $n \geq 16$**. Since** $14 \leq n-2 < n$**,**

$$\begin{aligned}
[S_{14} \wedge \ldots \wedge S_n] &\implies S_{n-2} \\
&\implies \exists\, k, l \in \mathbb{N} \ni n-2 = k\times 3 + l\times 8 \\
&\implies \exists\, k, l \in \mathbb{N} \ni n+1 = (k+1)\times 3 + l\times 8 \\
&\iff S_{n+1}\,.
\end{aligned}$$

This proves the induction step. QED