

# Relations

Recall this

**Definition:** The **cartesian Product** of two sets  $A, B$  is the set of all ordered pairs:

$$A \times B \stackrel{\text{def}}{=} \{(a, b) : a \in A, b \in B\} .$$

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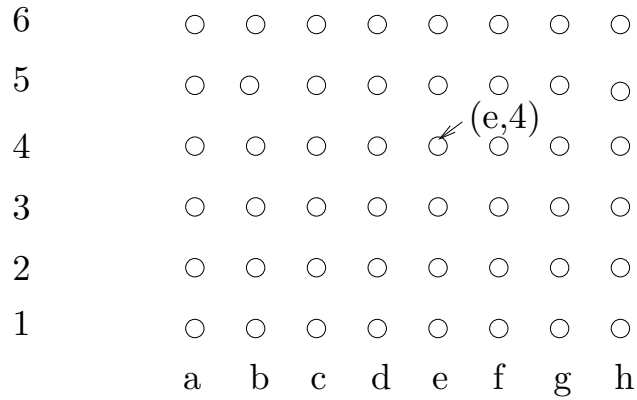
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**Example:** If  $|A| = m$  and  $|B| = n$  then the size of  $A \times B$  is

$$|A \times B| = mn = |A| \cdot |B| .$$

$$B = \{1, 2, 3, 4, 5, 6\}$$

↓

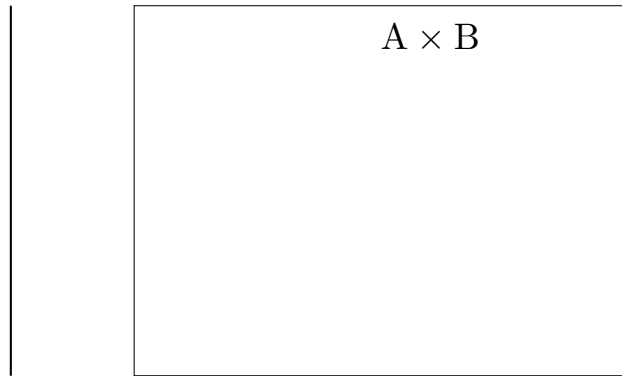


$$A = \{a, b, c, d, e, f, g, h\}$$

Picture of  $A \times B$

---

B



A

Picture of  $A \times B$

**Example:** Let  $A, B \subseteq \mathcal{U}$ . Then

$$A \times B = B \times A \iff [A = \emptyset \vee B = \emptyset \vee A = B].$$

**Proof of  $\Leftarrow$ :** Obvious.

**Proof of  $\Rightarrow$ :**

$$A \times B = B \times A \rightarrow [A = \emptyset \vee B = \emptyset \vee A = B]$$

is logically equivalent to

$$[A \times B = B \times A \wedge A \neq \emptyset \wedge B \neq \emptyset] \rightarrow A = B,$$

so we prove that this is a tautology.

**To prove:** Under assumptions  $[\dots]$ ,  $A \subseteq B$  and  $B \subseteq A$ .

**To prove:**  $\forall a \in \mathcal{U} a \in A \implies a \in B$  etc.

Let  $a \in A$ . Since  $B \neq \emptyset$ ,  $\exists b \in B$ . Then  $(a, b) \in A \times B$ . Since  $A \times B = B \times A$ ,  $(a, b) \in B \times A$ . Thus  $a \in B$ , and  $A \subseteq B$ .  $B \subseteq A$  is proved similarly. Hence  $A = B$ . **QED**

**Definition:** A relation from  $A$  to  $B$  is a subset of  $A \times B$ .

**Convention:** Let  $\mathcal{R} : A \rightarrow B$  be a relation. Instead of  $(a, b) \in \mathcal{R}$  we often write

$$a\mathcal{R}b$$

and say  $a$  is  $\mathcal{R}$ -related to  $b$ .

**Examples:** a) Let  $M$  =all male US-persons,  $F$  =all female US-persons.

$a_1)$

$$\mathcal{M} \stackrel{\text{def}}{=} \{(m, f) \in M \times F : m \text{ is married to } f\}$$

is an example. For  $(m, f) \in \mathcal{M}$  we write  $m\mathcal{M}f$  and say  $m$  is married to  $f$ .

$a_2)$

$$\mathcal{P} \stackrel{\text{def}}{=} \{(m, f) \in M \times F : m \text{ is a parent of } f\}$$

is another example. For  $(m, f) \in \mathcal{P}$  we write

$m\mathcal{P}f$  and say  $m$  is a father of  $f$  or  $f$  is a daughter of  $m$ .

b) Let  $A = B = \text{all US-persons}$ .

$b_1)$

$\mathcal{H} \stackrel{\text{def}}{=} \{(a, b) \in A \times A : a \& b \text{ have same hair color}\}$

is an example. For  $(a, b) \in \mathcal{H}$  we write  $a\mathcal{H}b$ .

This is a **Relation on  $A$** .

$b_2)$   $\mathcal{P} \stackrel{\text{def}}{=} \{(a, b) \in A \times A : a \text{ is a parent of } b\}$

is another example. For  $(a, b) \in \mathcal{P}$  we write  $a\mathcal{P}b$  and say  $a$  is a parent of  $b$  or  $b$  is a child of  $a$ . This is another relation on  $A$ .

c) Let  $A = B = \mathbb{Q}$ .

$c_1)$   $\leq \stackrel{\text{def}}{=} \{(a, b) \in \mathbb{Q} \times \mathbb{Q} : a \leq b\}$

is an example, **the usual order**. For  $(a, b) \in \leq$

we write  $a \leq b$  and say  $a$  is less than  $\langle$ or equal to $\rangle b$ .

$$c_2) \quad \geq \stackrel{\text{def}}{=} \{(a, b) \in \mathbb{Q} \times \mathbb{Q} : a \geq b\}$$

is another example. For  $(a, b) \in \geq$  we write  $a \geq b$  and say  $a$  is greater than  $\langle$ or equal to $\rangle b$ . This is again a relation on  $\mathbb{Q}$ .

d) Let  $A = B = \mathbb{Z}$ , fix a  $p \in \mathbb{Z}$ ,  $p > 0$ , and set

$$\equiv_p \stackrel{\text{def}}{=} \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \equiv b \pmod{p}\} .$$

This is an important relation on  $\mathbb{Z}$ .

**Example:** If  $|A| = m$  and  $|B| = n$  then the number of relations  $A \rightarrow B$  is

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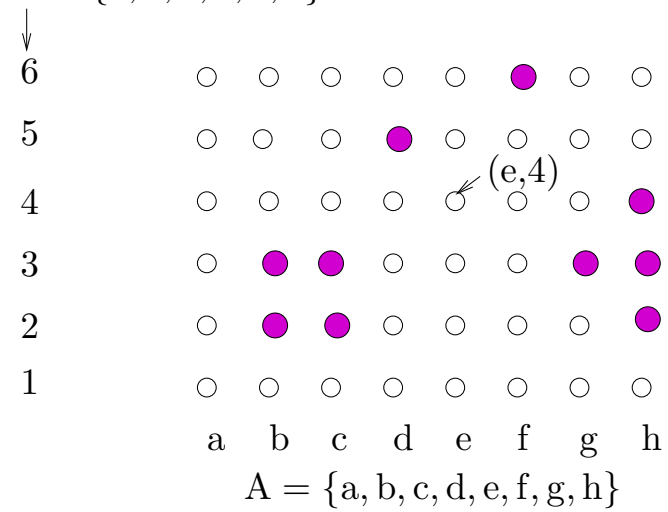
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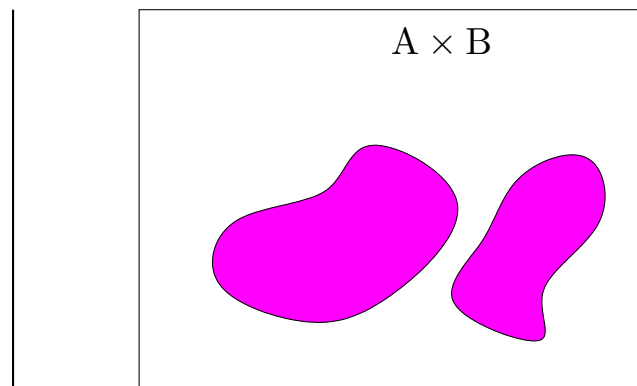
$$|\mathcal{P}[A \times B]| = 2^{mn} .$$

$$B = \{1, 2, 3, 4, 5, 6\}$$




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B



A

Picture of a Relation  $\mathcal{R}$

# The Transpose

**Definition:** The **Transpose** of a relation  $\mathcal{R} : A \rightarrow B$  is the relation

$$\mathcal{R}^T \stackrel{\text{def}}{=} \{(b, a) \in B \times A : (a, b) \in \mathcal{R}\} .$$

from  $B$  to  $A$ .

**Example:** The transpose of the relation  $\mathcal{F} \subset \mathcal{M} \times \mathcal{P}$ : “F is the father of C” is the relation

$\mathcal{C} \subset \mathcal{P} \times \mathcal{M}$ : “C is the child of F.”

[Here  $\mathcal{P}$  is “all people,”  $\mathcal{M}$  is “all males.”]

# Functions

**Definition:** A relation  $f$  from  $A$  to  $B$  is a **Function** if

$$\forall a \in A \exists! b \in B \ni \underbrace{(a, b) \in f}_{\substack{b=f(a) \\ afb}} .$$

$A$  is called the **Domain** and  $B$  the **Codomain** of  $f$ .

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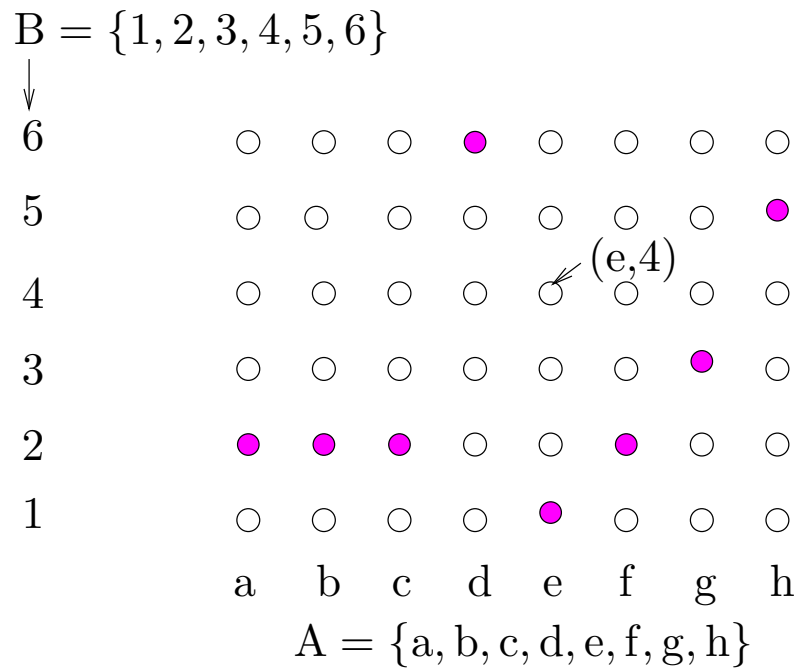
**Example:** If  $|A| = m$  and  $|B| = n$  then the number of functions  $A \rightarrow B$  is

$$n^m = |B|^{|A|} .$$

**Exercise:** Two functions  $f_1, f_2 : A \rightarrow B$  are equal if and only if

$$\forall a \in A \quad f_1(a) = f_2(a) .$$

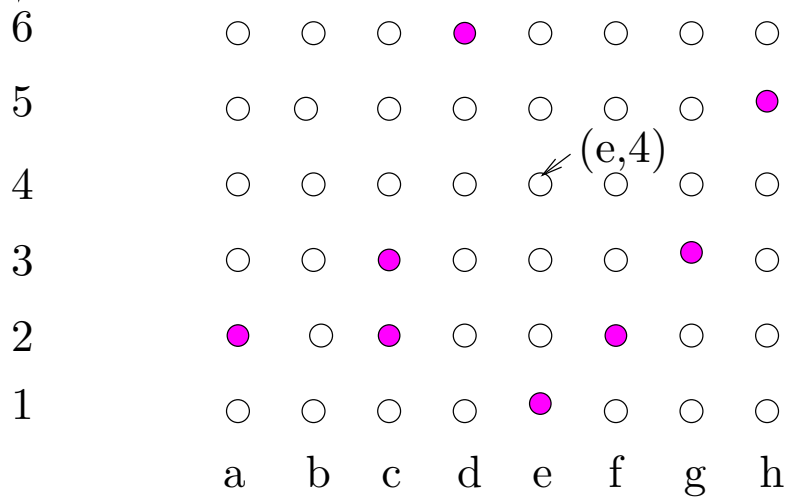
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Picture of a function  $f : A \rightarrow B$

$$B = \{1, 2, 3, 4, 5, 6\}$$

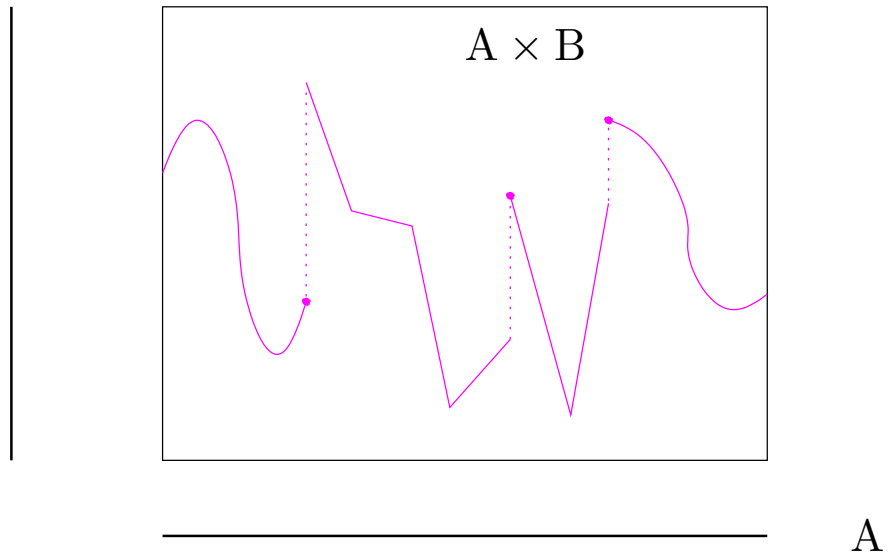
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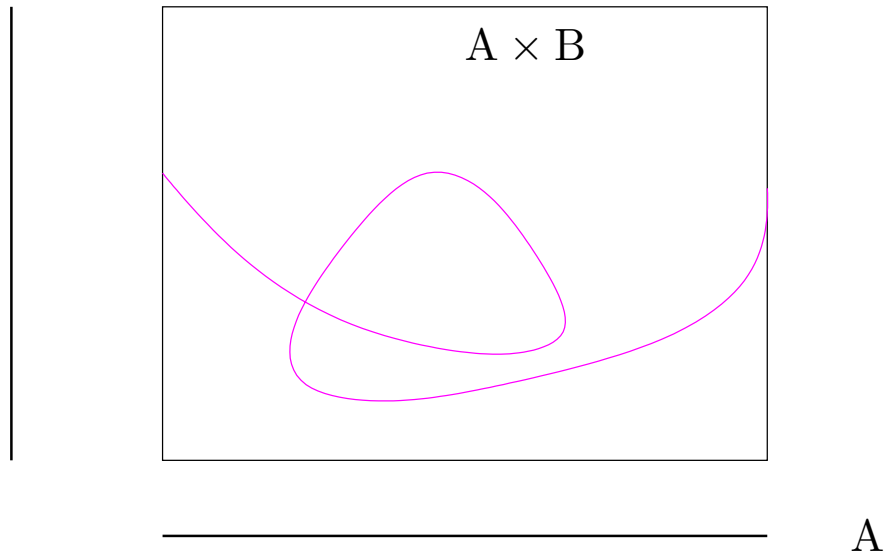
Picture of a function  $f : A \rightarrow B$ ?

B

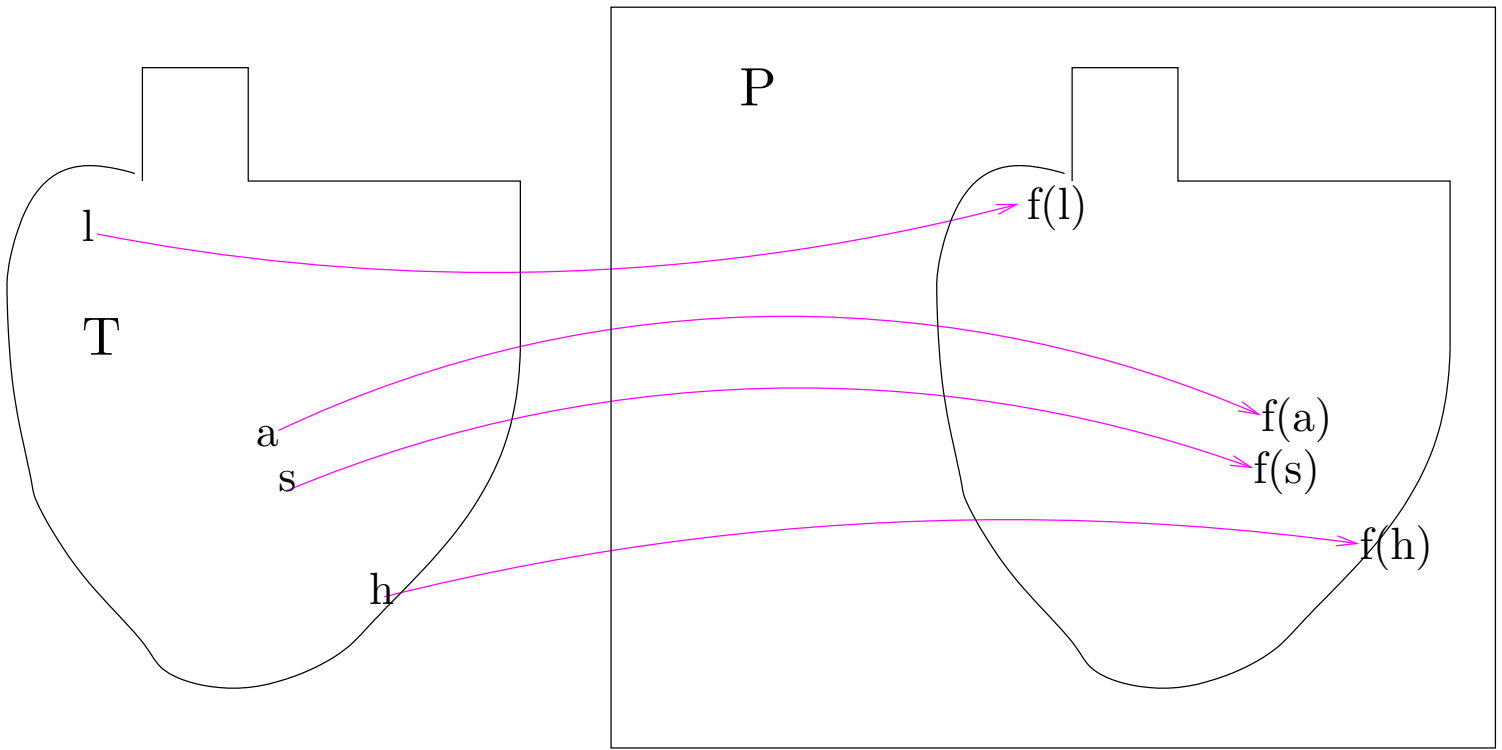


Picture of a function  $f: A \rightarrow B$

B



Picture of a function  $f : A \rightarrow B$ ?



Picture of a function or map  $f : A \rightarrow B$

**Definition:** Let  $f : A \rightarrow B$  be a function.

a) For  $A_0 \subseteq A$

$$f(A_0) \stackrel{\text{def}}{=} \{f(a) : a \in A_0\}$$

is the **<direct>** or **<forward>** Image of  $A_0$  under  $f$ .

The image  $f(A)$  of the whole domain is called the **Range of  $f$** .

b) For  $B_0 \subseteq B$

$$f^{-1}(B_0) \stackrel{\text{def}}{=} \{a \in A : f(a) \in B_0\}$$

is the **pre-image** or **inverse image** of  $B_0$  under  $f$ .

$$f^{-1}(B) = ?$$

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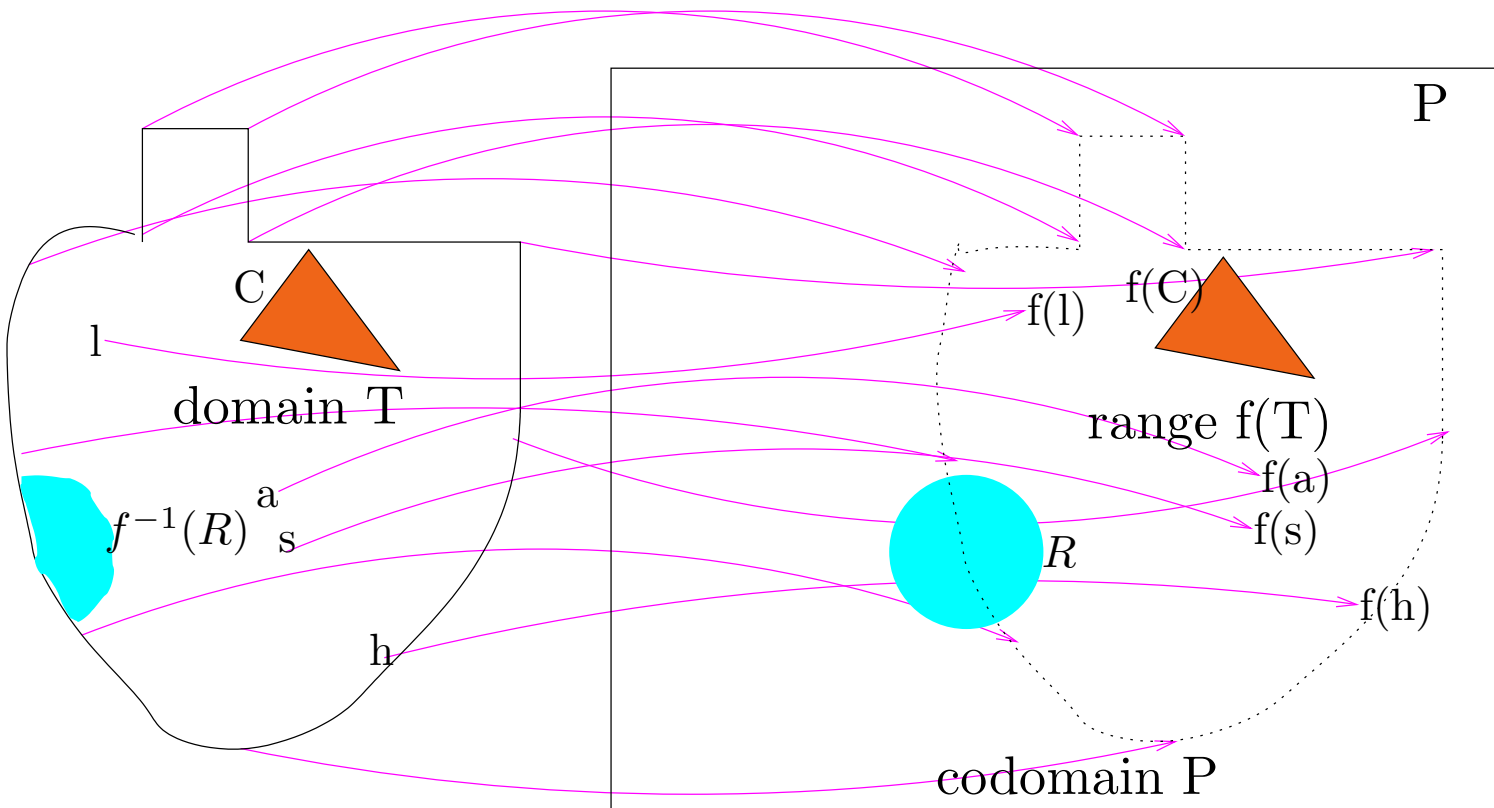
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$$f^{-1}(B) = ?$$

$$f^{-1}(B) = A .$$



Picture of a function or map  $f : A \rightarrow B$

**Definition:** A function  $f : A \rightarrow B$  is **1-1**, or **one-to-one** or **injective** if

$$\forall a, \bar{a} \in A \quad f(a) = f(\bar{a}) \implies a = \bar{a} .$$

In other words,  $f$  is injective if different elements  $a, \bar{a}$  of the domain have different images  $f(a), f(\bar{a})$  under  $f$ .

**Example:** If  $|A| = m$  and  $|B| = n$  then the number of injective functions  $A \rightarrow B$  is

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**Example:** If  $|A| = m$  and  $|B| = n$  then the number of injective functions  $A \rightarrow B$  is

$$\begin{array}{ll} 0 & \text{if } m > n, \\ \frac{n!}{(n-m)!} & \text{if } m \leq n, \end{array}$$

because

$$\underbrace{n(n-1)(n-2)\dots(n-m+1)}_{m \text{ factors}} = \frac{n!}{(n-m)!}$$

**Examples:**

- a) The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^2$  is not injective.
- b) The function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  given by  $f(x) = x^2$  is injective.
- c) The function  $f : x \mapsto x^3$  on  $\mathbb{R}$  is injective.

**Theorem:** Let  $f : A \rightarrow B$  be a function.  
 Then  $\forall A_1, A_2 \subseteq A$  and  $\forall B_1, B_2 \subseteq B$

$$B_1 \subseteq B_2 \implies f^{-1}(B_1) \subseteq f^{-1}(B_2) ;$$

$$A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2) ;$$

$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2) ;$$

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2) ;$$

$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2) . \text{ And}$$

$$f(A_1 \cap A_2) = f(A_1) \cap f(A_2) \text{ iff } f \text{ is 1-1 .}$$

**Proof:** The first line:  $a \in f^{-1}(B_1) \implies f(a) \in B_1$   
 $\implies f(a) \in B_2 \implies a \in f^{-1}(B_2)$ .

**line 2:**  $b \in f(A_1) \implies \exists a \in A_1 \ni f(a) = b \implies \exists a \in A_2 \ni f(a) = b \implies b \in f(A_2)$ .

**line 4:**  $f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2)$  follows from line 1. For the converse containment, let  $a \in f^{-1}(B_1) \cap f^{-1}(B_2)$ . Then  $f(a) \in B_1$  and  $f(a) \in B_2$ , thus  $f(a) \in B_1 \cap B_2$  and  $a \in f^{-1}(B_1 \cap B_2)$ .

**The last line:**  $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$  follows from line 2. Let us prove first the converse containment under the assumption that  $f$  is injective.

Let  $b \in f(A_1) \cap f(A_2)$ . Then

$$\begin{aligned} \exists a_1 \in A_1 \ni f(a_1) = b \quad \text{and} \\ \exists a_2 \in A_2 \ni f(a_2) = b . \end{aligned}$$

**As  $f$  is injective,  $a_1 = a_2 \in A_1 \cap A_2$ . Hence  $b = f(a_1) \in f(A_1 \cap A_2)$ .**

It is left to prove that

$$\{ \forall A_1, A_2 \subseteq A [ f(A_1 \cap A_2) = f(A_1) \cap f(A_2) ] \} \\ \implies f \text{ is injective.}$$

That is to say, we must show that under the assumption  $\{\dots\}$

$$\forall a_1, a_2 \in A [ f(a_1) = f(a_2) \implies a_1 = a_2 ] .$$

So **let**  $a_1, a_2 \in A$  **with**  $f(a_1) = f(a_2)$   
and **set**  $A_1 \stackrel{\text{def}}{=} \{a_1\}$  and  $A_2 \stackrel{\text{def}}{=} \{a_2\}$ .

Then  $f(A_1) = f(A_2) = \{b\}$   
and thus  $f(A_1) \cap f(A_2) \neq \emptyset$ .

Therefore  $f(A_1 \cap A_2) \neq \emptyset$   
and consequently  $A_1 \cap A_2 \neq \emptyset$ .

This means  $a_1 = a_2$ :  $f$  is indeed injective.

**QED**

**Definition:** Let  $f : A \rightarrow B$  be a function and  $A_0 \subseteq A$ . The **Restriction**  $f|_{A_0} : A_0 \rightarrow B$  is defined by

$$f|_{A_0} : a \mapsto f(x) \quad \forall a \in A_0 .$$

Conversely, let  $f : A \rightarrow B$  be a function and  $\bar{A} \supseteq A$  a **Superset of  $A$** . Then a function  $\bar{f} : \bar{A} \rightarrow B$  is an **Extension of  $f$  to  $\bar{A}$**  if

$$\bar{f}(a) = f(a) \quad \forall a \in A .$$